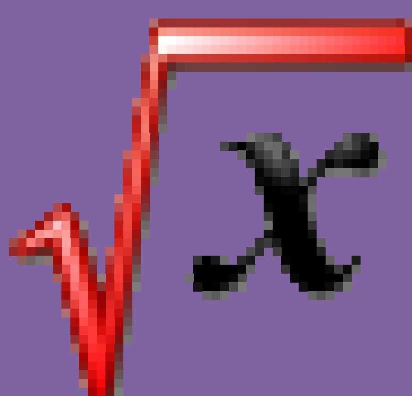
Teaching Some Selected Topics of

Based on National Mathematics Curriculum



Ethiopian Teachers Association (ETA)

Table of Content

Learning skills and remembering facts in mathematics are important but they are only the means to an end. Facts and skills are not important in themselves. They are important when we need them to solve a problem. Students will remember facts and skills easily when they use them to solve real problems. As well as using mathematics to solve real-life problems, students should also be taught about the different parts of mathematics, and how they fit together.

Mathematics can be taught using a step-by-step approach to a topic but it is important to show that many topics are linked. It is also important to show students that mathematics is done all over the world.

Teaching Methods in the Mathematics Classroom

This manual is all about the different ways you can teach a topic in the classroom. As you know, students learn things in many different ways. They don't always learn best by sitting and listening to the teacher. Students can learn by:

- Practising skills on their own
- Discussing mathematics with each other
- Playing mathematical games
- Doing puzzles
- Doing practical work
- Solving problems
- Finding things out for themselves.

In the classroom, students need opportunities to use different ways of learning. Using a range of different ways of learning has the following benefits:

- It motivates students
- It improves their learning skills
- It provides variety
- It enables them to learn things more quickly.

In this particular manual, we will look at some active methods of teaching Mathematics in relation to some selected topics of Grade 9 Mathematics.

About the Manual

This manual is not simply a collection of teaching ideas and activities. It describes an approach to teaching and learning mathematics. It also includes some teaching notes based on the currently revised Grade 9 National Mathematics curriculum. This manual can be best used as part of an approach to teaching using a plan or scheme of work to guide your teaching. It can also be used as a resource to help you with ideas for developing activities and selecting teaching methods to meet the needs of all pupils and to raise standards of achievement. Keep in mind that this manual can’t replace the teacher’s guide that accompanies the Grade 9 Mathematics student textbook. But, it will be a supplementary guide to those teachers who would wish to use it.

Key Features of the Manual

The topics are selected from six of the seven units of Grade 9 Mathematics curriculum. At the beginning of each topic, the competences of the topic are written to remind you about what you will expect from your students after teaching the specific topic. The methods suggested here in this manual are just suggestions and shall not be taken as prescriptions in any way. Selected teaching notes are also given using variety of examples. At the end of each topic, some exercises are given for you to practice and then give them to your students.

Topic 1: Expressing Repeating Decimals as Fractions

Competencies

- Express repeating decimals as fraction
- Show that repeating decimals are also rational numbers

Suggested ways of teaching this topic: Presentation and Explanation by the Teacher

This is a formal teaching method which involves the teacher presenting and explaining mathematics to the whole class. It can be difficult because you have to ensure that all students understand. This can be a very effective way of:

- Teaching a new piece of mathematics to a large group of students
- Drawing together everyone's understanding at certain stages of a topic
- Summarizing what has been learnt

As a brain storming activity you may ask students to

- List the types of decimals they know
- Give some examples of repeating decimals

Then, you might start with a question like:

'How can I express $4.1\overline{256}$ as a fraction?'

Then, using examples, you can show them how!

Lesson Notes

Let $x = 4.1\overline{256}$

Multiplying the above equation by 1000(where there are 3 repeating decimals), we can have:

$1000x = 4125.6\overline{256}$

Subtracting x from 1000x, we get

$$1000x - x = 4121.5$$
$$999x = 4121.5$$
$$x = \frac{4121.5}{999}$$

$$
\begin{aligned}
&- \frac{4121 - \frac{1}{2}}{999} \\
&= \frac{\frac{8243}{2}}{999} \\
&= \frac{8243}{1998} = 4.1\overline{256}
\end{aligned}
$$

Example 1:

Find the fraction represented by the repeating decimal $0.\overline{7}$.

Let *n* stand for $0.\overline{7}$or 0.77777 … So 10 *n* stands for $7.\overline{7}$or 7.77777 …

10 *n* and *n* have the same fractional part, so their difference is an integer.

$$
\begin{aligned}
10n &- 7.\overline{7} \\
- \; n &= 0.\overline{7} \\
\hline
9n &= 7
\end{aligned}
$$

You can solve this problem as follows.

$$
\begin{aligned}
9n &= 7 \\
n &= \frac{7}{9}
\end{aligned}
$$

So, $0.\overline{7} = \frac{7}{9}$

Example 2:

Find the fraction represented by the repeating decimal $0.\overline{36}$.

Let *n* stand for $0.\overline{36}$or 0.363636 … So 10 *n* stands for $3.6\overline{36}$or 3.63636 … and 100 *n* stands for $36.\overline{36}$or 36.3636 …

100 *n* and *n* have the same fractional part, so their difference is an integer. (The repeating parts are the same, so they subtract out.)

$$
\begin{aligned}
100n &= 36.\overline{36} \\
- \quad n &= \;\; 0.\overline{36} \\
\hline
99n &= 36
\end{aligned}
$$

You can solve this equation as follows:

$$99n = 35$$

$n = \frac{36}{99}$Now simplify $\frac{36}{99}$to $\frac{4}{11}$. So $0.\overline{36} = \frac{4}{11}$

Example 3:

Find the fraction represented by the repeating decimal $0.5\bar{4}$.

Let *n* stand for $0.5\bar{4}$ or 0.544444 … So 10 *n* stands for $5.\bar{4}$ or 5.444444 … and 100 *n* stands for $54.\bar{4}$ or 54.4444 …

Since 100 *n* and 10 *n* have the same fractional part, their difference is an integer. (Again, notice how the repeated parts must align to subtract out.)

$$\begin{aligned} 100n &= 54.\bar{4} \\ -10n &= 5.\bar{4} \\ \hline 90n &= 49 \end{aligned}$$

You can solve this equation as follows.

$$90n = 49$$
$$n = \frac{49}{90}$$

So, $0.5\bar{4} = \frac{49}{90}$

Concluding Activities

Make sure that students arrive at the following conclusive statements

Every decimal numeral which is either

- A terminating decimal number, or
- A repeating non terminating decimal number, can be expressed as a fraction.

And conversely, every fractional number represents a terminating, or repeating non – terminating decimal number.

Practice Exercises

Find the rational number represented by each of the following:

- $0.1\dot{5}$
- $0.7\overline{14}$
- $3.\overline{271}$
- $0.\overline{14285}$

Topic 2: Locating Irrational Numbers on a Number Line

Competencies

- Identify irrational numbers
- Locate some irrational numbers on a number line.

Suggested ways of teaching this topic: Games and Presentation of the teacher

Using games can make mathematics classes very enjoyable, exciting and interesting. Mathematical games provide opportunities for students to be actively involved in learning. Games allow students to experience success and satisfaction, thereby building their enthusiasm and self-confidence. But mathematical games are not simply about fun and confidence building. Games help students to:

- Understand mathematical concepts
- Develop mathematical skills
- Know mathematical facts
- Learn the language and vocabulary of mathematics
- Develop ability in mental mathematics.

Game: Locate Irrational Numbers

(A game for two players)

One player chooses an irrational number to be located in between two consecutive integers by the other. The player who locates the number quickly wins the game.

As a starter activity, you may ask students to play the above game and then for consolidation and practice you may give them some irrational numbers to be located in between two integers by the students.

Lesson Notes

Example:

Locate negative square root of 14 between two consecutive integers on the number line.

Forget the negative for a moment. What two numbers does the square root of 14 come between? As the square root of 9 is 3 and the square root of 16 is 4, the square root of 14 is somewhere between 3 and 4. Since the question says negative square root, it is between -3 and -4.

Practice Exercises

Locate the following irrational numbers between two consecutive integers on the number line.

a) square root of 21

b) negative square root of 75
c) negative square root of 118
d) square root of 232
e) square root of 600

Topic 3: Simplification of Radicals

Competencies

- Classify simplified radicals as rational or irrational
- Distinguish between like terms and unlike terms
- Use properties and theorems in simplifying radicals

Suggested ways of teaching this topic: Explanation by the Teacher and Practice of Students

Starter Activities

The teacher may start with some brainstorming questions like:

"When do we say that a radical is simplified or is in its simplest form?"

Expected Answers:

- When the radicand has no square factors.
- A radical is also in simplest form when the radicand is not a fraction.

Then, the teacher may explain "To put a radical expression in its simplest form, we make use of the following theorem:"

Theorem: For all non-negative real numbers a and b, $\sqrt{a} \times \sqrt{b} = \sqrt{a \times b}$

Here is a simple illustration: $\sqrt{100} = \sqrt{4 \times 25} = \sqrt{4} \times \sqrt{25} = 2 \times 5 = 10$

$$\sqrt{18} = \sqrt{9 \cdot 2} = \sqrt{9} \cdot \sqrt{2} = 3\sqrt{2}.$$

Therefore, we have simplified $\sqrt{18}$.

Let students practice the following problems themselves first!

a) $\sqrt{28} = \sqrt{4 \cdot 7} = \sqrt{4} \cdot \sqrt{7} = 2\sqrt{7}$

b) $\sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25} \cdot \sqrt{2} = 5\sqrt{2}$

c) $\sqrt{45} = \sqrt{9 \cdot 5} = \sqrt{9} \cdot \sqrt{5} = 3\sqrt{5}$

d) $\sqrt{98} = \sqrt{49 \cdot 2} = 7\sqrt{2}$

e) $\sqrt{48} = \sqrt{2 \cdot 2 \cdot 2 \cdot 2 \cdot 3} = 4\sqrt{3}$

f) $\sqrt{300} = \sqrt{100 \cdot 3} = 10\sqrt{3}$

g) $\sqrt{150} = \sqrt{25 \cdot 6} = 5\sqrt{6}$

h) $\sqrt{80} = \sqrt{16 \cdot 5} = 4\sqrt{5}$

Example: Reduce the following to their lowest terms.

a) $\dfrac{\sqrt{20}}{2} = \dfrac{\sqrt{4 \cdot 5}}{2} = \dfrac{2\sqrt{5}}{2} = \sqrt{5}$

b) $\dfrac{\sqrt{72}}{3} = \dfrac{\sqrt{36 \cdot 2}}{3} = \dfrac{6\sqrt{2}}{3} = 2\sqrt{2}$

c) $\dfrac{\sqrt{22}}{2}$ = The radical is in its simplest form. The fraction cannot be reduced.

What are similar radicals?

Similar radicals have the same radicand. We add them as like terms.

$$7 + 2\sqrt{3} + 5\sqrt{2} + 6\sqrt{3} - \sqrt{2} = 7 + 8\sqrt{3} + 4\sqrt{2}.$$

$2\sqrt{3}$ and $6\sqrt{3}$ are similar, as are $5\sqrt{2}$ and $-\sqrt{2}$. We combine them by adding their coefficients.

Lesson Notes

Examples: Simplify each radical, and then add the similar radicals.

a) $\sqrt{18} + \sqrt{8} = 3\sqrt{2} + 2\sqrt{2} = 5\sqrt{2}$

b) $4\sqrt{75} - 2\sqrt{147} + \sqrt{3} = 4\sqrt{25 \cdot 3} - 2\sqrt{49 \cdot 3} + \sqrt{3}$

$= 4 \cdot 5\sqrt{3} - 2 \cdot 7\sqrt{3} + \sqrt{3}$

$= 20\sqrt{3} - 14\sqrt{3} + \sqrt{3} = 7\sqrt{3}$

c) $3\sqrt{28} + \sqrt{88} - 2\sqrt{112} = 3\sqrt{4 \cdot 7} + \sqrt{4 \cdot 22} - 2\sqrt{16 \cdot 7}$

$= 3 \cdot 2\sqrt{7} + 2\sqrt{22} - 2 \cdot 4\sqrt{7}$

$= 6\sqrt{7} + 2\sqrt{22} - 8\sqrt{7}$

$$= 2\sqrt{22} - 2\sqrt{7}$$

d) $3 + \sqrt{24} + \sqrt{54} = 3 + \sqrt{4\cdot 6} + \sqrt{9\cdot 6}$

$= 3 + 2\sqrt{6} + 3\sqrt{6}$

$= 3 + 5\sqrt{6}$

e) $1 - \sqrt{128} + \sqrt{18} = 1 - \sqrt{64\cdot 2} + \sqrt{9\cdot 2}$

$= 1 - 8\sqrt{2} + 3\sqrt{2}$

$= 1 - 5\sqrt{2}$

Examples: Simplify the following.

a) $\dfrac{4 - \sqrt{8}}{2} = \dfrac{4 - 2\sqrt{2}}{2} = 2 - \sqrt{2}$, On dividing each term in the numerator by 2.

b) $\dfrac{10 + \sqrt{50}}{5} = \dfrac{10 + 5\sqrt{2}}{5} = 2 + \sqrt{2}$

c) $\dfrac{6 + \sqrt{24}}{6} = \dfrac{6 + 2\sqrt{6}}{6} = \dfrac{3 + \sqrt{6}}{3}$ Dividing each term by 2.

Concluding Activities

Help students to conclude that we say that a radical is simplified or is in its simplest form when the radicand:

- Have no square factors.
- Is not a fraction

Make sure that students are capable of classifying simplified radicals as rational or irrationals, distinguishing between like terms and unlike terms and they can use properties and theorems in simplifying radical expressions allowing them to practice more.

Practice Exercises

1. Simplify

a)	b)	c)	d)
$\sqrt{3}\times\sqrt{21}$	$\frac{5\sqrt{3}}{8\sqrt{5}}$	$\frac{\sqrt{5}\times\sqrt{10}}{5}$	$\sqrt{5}(\sqrt{3}+1)$
e)	f)	g)	h)
$\frac{5\sqrt{14}}{3\sqrt{2}}$	$\frac{15\sqrt{24}}{5\sqrt{3}}$	$\frac{3\sqrt{5}}{5\sqrt{3}}$	$\frac{3\sqrt{12}}{\sqrt{3}}$

2. Simplify

a) $\sqrt{89}+\sqrt{26}$

b) $\sqrt{54}-3\sqrt{6}$

c) $3\sqrt{6}\left(2\sqrt{12}+4\right)$

d) $2\sqrt{7}-\frac{1}{\sqrt{7}}$

e) $\sqrt{32}-\sqrt{50}+\sqrt{18}$

f) $2\times\sqrt{\frac{1}{2}}\times6\times\sqrt{48}$

g) $3\sqrt{12}-\sqrt{3}+2\sqrt{\frac{1}{3}}-\sqrt{27}$

h) $\left(5\sqrt{6}-7\sqrt{2}\right)\left(2\sqrt{6}+3\sqrt{2}\right)$

Topic 4: Rationalization of Radicals

Competencies

- Explain the notion of rationalization.
- Identify a rationalizing factor for a given expression.

Suggested ways of teaching this topic: Explanation by the Teacher and Practice of Students

Starter Activities: The teacher may start with revising the rules for multiplying and dividing radicals using examples:

The rules are:

1. $\sqrt{a}\cdot\sqrt{b} = \sqrt{ab}$

2. $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$

Let students do the problems themselves first then the teacher might summarize.

Example 1: Multiply.

a) $\sqrt{3}\cdot\sqrt{5} = \sqrt{15}$

b) $2\sqrt{3}\cdot 3\sqrt{7} = 6\sqrt{42}$

c) $\sqrt{18}\cdot\sqrt{2} = \sqrt{36} = 6$

d) $(2\sqrt{5})^2 = 4\cdot 5 = 20$

e) $\sqrt{a-b}\,\sqrt{a-b} = \sqrt{a^2 - b^2}$ The difference of two squares

Example 2: Multiply, and then simplify:

a) $\sqrt{14x^5}\cdot\sqrt{7x^9}$

b) $(\sqrt{6} + \sqrt{2})(\sqrt{6} - \sqrt{2})$.

Solution:

1. $\sqrt{14x^5}\cdot\sqrt{7x^9} = \sqrt{2\cdot 7\cdot 7\cdot x^{14}} = 7x^7\sqrt{2}$

2. The student should recognize the *form* those factors will produce.

The difference of two squares

$$(\sqrt{6} + \sqrt{2})(\sqrt{6} - \sqrt{2}) = (\sqrt{6})^2 - (\sqrt{2})^2$$

$$= 6 - 2$$

$$= 4.$$

Practice and Consolidation is needed here! Thus, the teacher should guide students to recognize the forms of “difference of two squares” letting them practice on products of the following type.

Lesson Notes

Example 3: Multiply.

a) $(\sqrt{5} + \sqrt{3})(\sqrt{5} - \sqrt{3}) = 5 - 3 = 2$

b) $(2\sqrt{3} + \sqrt{6})(2\sqrt{3} - \sqrt{6}) = 4 \cdot 3 - 6 = 12 - 6 = 6$

c) $(1 + \sqrt{x - 1})(1 - \sqrt{x + 1}) = 1 - (x + 1) = 1 - x - 1 = -x$

d) $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b$

Example 4: Given $(x - 1 - \sqrt{2})\,(x - 1 + \sqrt{2})$

a) What form does that produce?

Expected Answer:

The difference of two squares of the form $a^2 - b^2$, where $(x - 1)$ is "*a*" and $\sqrt{2}$ is "*b*"

b) Find the final product.

$$(x - 1 - \sqrt{2})(x - 1 + \sqrt{2}) = (x - 1)^2 - 2$$

$$= x^2 - 2x + 1 - 2,$$ On squaring the binomial,

$$= x^2 - 2x - 1$$

Example 5: Multiply.

$$(x + 3 + \sqrt{3})(x + 3 - \sqrt{3}) = (x + 3)^2 - 3$$

$$= x^2 + 6x + 9 - 3$$

$$= x^2 + 6x + 6$$

Example 6: Simplify the following.

a) $\frac{\sqrt{6}}{\sqrt{3}} = \sqrt{2}$ b) $\frac{6\sqrt{10}}{8\sqrt{2}} = \frac{3}{4}\sqrt{5}$ c) $\frac{a\sqrt{a^3}}{\sqrt{a}} = a\sqrt{a^2} = a \cdot a = a^2$< TR>

What are Conjugate pairs?

The conjugate of $a + \sqrt{b}$ is $a - \sqrt{b}$. They are a conjugate pair.

Example 7: Multiply $6 - \sqrt{2}$ with its conjugate.

Solution: The product of a conjugate pair is the difference of two squares

$$(6 - \sqrt{2})(6 + \sqrt{2}) = 36 - 2 = 34$$

CONCLUSION

When we multiply a conjugate pair, the radical vanishes and we obtain a rational number. This process is called Rationalization.

Example 8: Multiply each number with its conjugate.

1. $x + \sqrt{y} = (x + \sqrt{y})(x - \sqrt{y}) = x^2 - y$
2. $2 - \sqrt{3} = (2 - \sqrt{3})(2 + \sqrt{3}) = 4 - 3 = 1$
3. $\sqrt{6} + \sqrt{2}$

   This time, students should be able to write the product immediately: $6 - 2 = 4$

4. $4 - \sqrt{5} = 16 - 5 = 11$

Example 9: Rationalize the denominator of $\frac{1}{3+\sqrt{2}}$

Solution:

Multiply both the denominator and the numerator by the conjugate of the denominator; that is, multiply them by $3-\sqrt{2}$.

$$\frac{1}{3 + \sqrt{2}} = \frac{3 - \sqrt{2}}{9 - 2} = \frac{3 - \sqrt{2}}{7}$$

The numerator becomes $3-\sqrt{2}$. The denominator becomes the difference of the two squares.

Example 10: Rationalize the denominator of $\dfrac{\sqrt{3}}{\sqrt{3} + 2}$

Solution: $\dfrac{\sqrt{3}}{\sqrt{3} + 2} = \dfrac{\sqrt{3}(\sqrt{3} - 2)}{3 - 4} = \dfrac{\sqrt{3}\cdot\sqrt{3} - 2\sqrt{3}}{-1}$

$= -(3 - 2\sqrt{3})$

$= 2\sqrt{3} - 3$

Example 11: Write out the steps that show each of the following equivalences.

a) $\dfrac{1}{\sqrt{5} + \sqrt{3}} = ½(\sqrt{5} - \sqrt{3})$

$\dfrac{1}{\sqrt{5} + \sqrt{3}} = \dfrac{\sqrt{5} - \sqrt{3}}{5 - 3} = \dfrac{\sqrt{5} - \sqrt{3}}{2} = ½(\sqrt{5} - \sqrt{3})$

b) $\dfrac{2}{3 + \sqrt{5}} = ½(3 - \sqrt{5})$

$\dfrac{2}{3 + \sqrt{5}} = \dfrac{2(3 - \sqrt{5})}{9 - 5} = \dfrac{2(3 - \sqrt{5})}{4} = ½(3 - \sqrt{5})$

c) $\dfrac{7}{3\sqrt{5} + \sqrt{3}} = \dfrac{3\sqrt{5} - \sqrt{3}}{6}$

$\dfrac{7}{3\sqrt{5} + \sqrt{3}} = \dfrac{7(3\sqrt{5} - \sqrt{3})}{9\cdot 5 - 3} = \dfrac{7(3\sqrt{5} - \sqrt{3})}{42} = \dfrac{3\sqrt{5} - \sqrt{3}}{6}$

d) $\dfrac{\sqrt{2} + 1}{\sqrt{2} - 1} = 3 + 2\sqrt{2}$

$\dfrac{\sqrt{2} + 1}{\sqrt{2} - 1} = \dfrac{(\sqrt{2} + 1)^2}{2 - 1} = 2 + 2\sqrt{2} + 1,$

Example 12: Simplify $\dfrac{1}{\dfrac{1}{\sqrt{5}}+\dfrac{1}{\sqrt{6}}}$

At every step of the solution the teacher is expected to ask the students to give reasons.

Solution: $\dfrac{1}{\dfrac{1}{\sqrt{5}}+\dfrac{1}{\sqrt{6}}} = \dfrac{1}{\dfrac{\sqrt{6}+\sqrt{5}}{\sqrt{5}\sqrt{6}}}$ Why?

$= \dfrac{\sqrt{5}\sqrt{6}}{\sqrt{6}+\sqrt{5}}$ Why?

$= \dfrac{\sqrt{5}\sqrt{6}(\sqrt{6}-\sqrt{5})}{6-5}$ Why?

$= 6\sqrt{5}-5\sqrt{6}$ Why? .

Example 13: Simplify $\dfrac{1}{\dfrac{1}{\sqrt{2}}-\dfrac{1}{\sqrt{3}}}$

$\dfrac{1}{\dfrac{1}{\sqrt{2}}-\dfrac{1}{\sqrt{3}}} = \dfrac{1}{\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{2}\sqrt{3}}}$ Why?

$= \dfrac{\sqrt{2}\sqrt{3}}{\sqrt{3}-\sqrt{2}}$ Why?

$= \dfrac{\sqrt{2}\sqrt{3}(\sqrt{3}+\sqrt{2})}{3-2}$ Why?

$= 3\sqrt{2}+2\sqrt{3}$ Why?

Concluding Activities

Make sure that the students summarize that rationalization of the denominator can be done by multiplying both the denominator and the numerator by the conjugate of the denominator. Check if every student has developed the skill of rationalizing radical expressions by giving them practice exercises.

Practice Exercises

1. Rationalize the following by multiplying with an appropriate conjugate and simplify

   a) $(\sqrt{5} - \sqrt{3})$

   b) $\sqrt{15} + \sqrt{12}$

2. Rationalize the denominator and simplify

   a) $\dfrac{5\sqrt{8}}{4\sqrt{3}}$

   b) $\dfrac{2}{\sqrt{5} - \sqrt{3}}$

   c) $\dfrac{\sqrt{3} + 4}{\sqrt{3} - 2}$

   d) $\dfrac{\sqrt{5}}{\sqrt{20} + \sqrt{15}}$

   e) $\dfrac{2 - 2\sqrt{3}}{9 - 5\sqrt{3}}$

   f) $\dfrac{\sqrt{10} - \sqrt{2}}{\sqrt{18}}$

   g) $\dfrac{12}{3\sqrt{2} - 2\sqrt{3}}$

   h) $\dfrac{1}{6}\sqrt{72} + \dfrac{1}{\sqrt{8}}$

Unit 2: Solution of Equations

Topic 1: Equations Involving Exponents

Competencies

- Solve equations involving exponents
- Use exponent rules

Suggested ways of teaching this topic: Teacher guided group discussion

Starter Activities

The teacher may start raising brainstorming questions of the following type:

- "For what value(s) of x will $2^x = 8$?"
- "When do we say $2^x = 2^y$?

And, let the students come up with answers discussing in pairs or in small groups.

Expected Answers:

- $x = 3$ for the first and $x = y$ for the second

The teacher should summarize students discussion raising the rules for exponents andthe fact that for a > 0, $a^x = a^y$ if and only if $x = y$.

Lesson Notes

Exponent Rules:

For a, b >0 and x and y are real numbers;

1. $a^x \times a^y = a^{x+y}$
2. $a^x \div a^y = a^{x-y}$
3. $(a^x)^y = (a^y)^x = (a)^{xy}$
4. $a^x \times b^x = (ab)^x$
5. $a^x \div b^x = (a \div b)^x$

Example1: Find the value of x

a) $4^{2x} = 32$

b) $3^{x+5} = 27$

Solution:

a) $4^{2x} = 32$
$(2^2)^{2x} = 2^5$
$2^{4x} = 2^5$
$4x = 5$
$x = 5/4$

b) $3^{x+5} = 27$
$3^{x+5} = 3^3$
$x + 5 = 3$
$x = 3 - 5$
$x = -2$

Example 2: Find the solution to the following equations

a) $(27)(9^{x-1}) = 81^x$ b) $2^{5-x} = (\frac{1}{4})^{x-3}$

Solution:

a) $(27)(9^{x-1}) = 81^x$
$(3^3)(3^{2(x-1)}) = 3^{4x}$
$3^{3+2x-2} = 3^{4x}$
$3^{2x+1} = 3^{4x}$
$2x+1 = 4x$
$-2x = -1$
$x = 1/2$

b) $2^{5-x} = (\frac{1}{4})^{x-3}$
$2^{5-x} = 2^{-2(x-3)}$
$2^{5-x} = 2^{-2x+6}$
$5 - x = -2x + 6$
$x = 1$

Example 3: Solve $(2t - 3)^{-2/3} = -1$.

Solution:

Raise each side to the power -3 to eliminate the root and the negative sign in the exponent:

$(2t - 3)^{-2/3} = -1$

$[(2t - 3)^{-2/3}]^{-3} = (-1)^{-3}$ Raise each side to the -3 power.

$(2x - 3)^2 = -1$ Multiply the exponents:

There is no real number which when squared can give – 1. Thus, the equation has no solution.

Concluding Activities

Make sure that students have got the idea, "To solve equations involving exponents:

- Use the exponent rules and
- The fact that $a^x = a^y$ if and only if $x = y$."

Practice Exercise

Solve the following equations

a) $4^{6x} = 32^{5x+3}$	b) $\frac{32^{2/5}}{4^{-5/2}} = 2^{-x}$	c) $\frac{3^{-\sqrt{5}}3^{\sqrt{5}}}{\sqrt{3}} = 3^{2x}$

Competencies

- Solve equations involving radicals
- Write solutions of equations involving radicals

Suggested ways of teaching this topic: Teacher guided group discussion

Starter Activities

The teacher may start raising brainstorming questions of the following type:

- "For what value(s) of x will $x^2 = 16$? ;$2^x = 16$?"
- What is the difference between $x^2 = 16$ and $2^x = 16$
- What is the difference between $x^2 = 16$ and $x = \sqrt{16}$?
- What do you think is the solution of $\sqrt{x} = 16$?

And, let the students come up with answers discussing in pairs or in small groups.

Expected Answers:

- $x^2 = 16$ for $x = 4$ and $x = -4$; $2^x = 16$ *for* $x = 4$ only
- *They have different solutions*
- $x^2 = 16$ for $x = 4$ and $x = -4$; $x = \sqrt{16}$ *for* $x = 4$ only. So, they have different solutions
- $\sqrt{x} = 16$ is true if $x = 16^2 = 256$.

The teacher then shall explain equations of the last form are radical equations and in this lesson students will solve equations that have the variable under a radical sign. At this stage, it is also better to inform students that they will be introduced to the concept of extraneous roots and see the necessity of checking all solutions by substituting them back into the original equation.

Lesson Notes

An equation that has a variable in a radicand is called a **radical equation**. The following are some examples of radical equations:

$$\sqrt{x+1} = 7 \;;\; 2\sqrt{x+1} - 5 = -1 \;;\; \sqrt{x+3} + \sqrt{2x} = x + 1 \;;\; \sqrt[3]{2x+3} = 5$$

To solve an equation having a term with a variable in a radicand, start by "isolating" such a term on one side of the equation. Then raise the expression on each side of the equal sign to a power equal to the index of the radical. This is shown in the examples below.

Example 1: Solve the following equation

$$\sqrt{x} + 3 = 7$$

Solution:

$\sqrt{x} + 3 = 7 \rightarrow \sqrt{x} = 7 - 3$

$\rightarrow \sqrt{x} = 4$

$\rightarrow$ *x = 16 (squaring both sides)*

Check! *(Substitute x = 16 in the original equation and see if it gives a true statement)*

$\sqrt{x} + 3 = 7 \rightarrow \sqrt{16} + 3 = 7$

$\rightarrow 4 + 3 = 7$ *which is true!*

Example 2: Solve: $4\sqrt{5x - 1} - 9 = 7$

Solution:

$4\sqrt{5x - 1} - 9 = 7 \rightarrow 4\sqrt{5x - 1} = 7 + 9$

$\rightarrow 4\sqrt{5x - 1} = 16$

$\rightarrow \sqrt{5x - 1} = 4$

$\rightarrow 5x - 1 = 16$ *(squaring both sides)*

$\rightarrow 5x = 17$

$\rightarrow x = 17/5$

Check!(Substitute x = 17/5 in the original equation and see if it gives a true statement)

$4\sqrt{5x-1}-9=7 \rightarrow 4\sqrt{5(\frac{17}{5})-1}-9=7$

$\rightarrow 4\sqrt{16}-9=7$

$\rightarrow$ 16 – 9 = 7; which is true!

The teacher may pose a question: "In Examples 1 and 2 above we checked our solution. We did not stress the necessity of checking our solutions. Why are we doing this now?"

Answer:

Actually, checking solutions is a good practice to follow regardless of the type of equation you are solving. We should have done it when solving these other equations too. However, it is especially important to check solutions when solving radical equations because of the process of squaring both sides. The process of squaring can introduce unacceptable or **extraneous roots**. Because squaring can introduce these extraneous roots, it is essential we check the solutions we find to any equation that involves squaring both sides. Let's see the following example:

Example 3: Solve: $\sqrt{4x+17}-3=x$

Solution:

$\sqrt{4x+17}-3=x \rightarrow \sqrt{4x+17}=x+3$

$\rightarrow 4x+17=x^2+6x+9$

$\rightarrow x^2+2x-8=0$

$\rightarrow x=-4$ *or* $x=2$

Check!(Substitute x = -4 and x =2 in the original equation and see which of these give atrue statement)

For $x=-4, \sqrt{4x+17}-3=x \rightarrow \sqrt{4(-4)+17}-3=-4$

$\rightarrow 1-3=-4$ *(This is wrong!)*

Thus, $x = -4$ *can't be a solution. (We call x = - 4 extraneous solution)*

For $x = 2$, $\sqrt{4x+17} - 3 = x \rightarrow \sqrt{4(2)+17} - 3 = 2$

$\rightarrow 5 - 3 = 2$ *((This is correct!)*

Thus, $x = 2$ *Is a solution.*

Multiple Radicals - *(Optional! The teacher could use this for brighter students)*

Sometimes, a radical equation contains more than one term with a variable in a radicand. When this happens, you have to "isolate and raise to a power" more than once. Generally speaking it is better to isolate the more complicated radical first, as this can simplify the process of raising the expressions to a power.

Example 1: Solve: $\sqrt{2x+5} - \sqrt{x-1} = 2$

Solution:

$$\sqrt{2x+5} - \sqrt{x-1} = 2$$
$$\sqrt{2x+5} = 2 + \sqrt{x-1}$$
$$\left(\sqrt{2x+5}\right)^2 = \left(2 + \sqrt{x-1}\right)^2$$
$$2x + 5 = 4 + 4\sqrt{x-1} + (x-1)$$
$$2x + 5 - 4 - x + 1 = 4\sqrt{x-1}$$
$$x + 2 = 4\sqrt{x-1}$$
$$(x+2)^2 = \left(4\sqrt{x-1}\right)^2$$
$$x^2 + 4x + 4 = 16(x-1)$$
$$x^2 + 4x + 4 = 16x - 16$$
$$x^2 - 12x + 20 = 0$$
$$(x-10)(x-2) = 0$$
$$x = 10 \quad \text{or} \quad x = 2$$

Check! (Substitute both x = 10 and x = 2 and see if you get true statements)

For x = 10,

$$\sqrt{2x+5} - \sqrt{x-1} = 2$$
$$\sqrt{2(10)+5} - \sqrt{(10)-1} = 2$$
$$\sqrt{25} - \sqrt{9} = 2$$
$$5 - 3 = 2 \quad \text{(true)}$$

For x = 2,

$$\sqrt{2x+5} - \sqrt{x-1} = 2$$
$$\sqrt{2(2)+5} - \sqrt{(2)-1} = 2$$
$$\sqrt{9} - \sqrt{1} = 2$$
$$3 - 1 = 2 \quad \text{(true)}$$

So, the solution is x = 10 and x = 3

Example 2: Solve: $\sqrt{3x-2}+\sqrt{x-1}=3$

Solution:

$$\sqrt{3x-2}+\sqrt{x-1}=3$$
$$\sqrt{3x-2}=3-\sqrt{x-1}$$
$$\left(\sqrt{3x-2}\right)^2=\left(3-\sqrt{x-1}\right)^2$$
$$3x-2=9-6\sqrt{x-1}+(x-1)$$
$$3x-2-9-x+1=-6\sqrt{x-1}$$
$$2x-10=-6\sqrt{x-1}$$
$$x-5=-3\sqrt{x-1}$$
$$(x-5)^2=\left(-3\sqrt{x-1}\right)^2$$
$$x^2-10x+25=9(x-1)$$
$$x^2-10x+25=9x-9$$
$$x^2-19x+34=0$$
$$(x-17)(x-2)=0$$
$$x=17 \quad \text{or} \quad x=2$$

Check!

$$\sqrt{3(17)-2}+\sqrt{(17)-1}=3$$
$$\sqrt{49}+\sqrt{16}=3$$
$$7+4=3 \quad \text{(false)}$$
So $x = 17$ is an extraneous root

$$\sqrt{3(2)-2}+\sqrt{(2)-1}=3$$
$$\sqrt{4}+\sqrt{1}=3$$
$$2+1=3 \quad \text{(true)}$$
So $x = 2$ is a solution.

Concluding Activities

Make sure that the students got the idea of solving radical equations asking them to tell you the general format of solving these types of equations. The students are expected to tell you the following summary:

Equation involving radicals can be solved by squaring both sides, to eliminate the radical, and then solve for the value of the variable. Always checking the value obtained on the original equation is important to avoid extraneous roots in writing the solution.

Practice Exercise

1. Solve each of the following equations:

1. $\sqrt{x}=4$
2. $\sqrt{x-3}=5$
3. $\sqrt{2x-1}=-3$
4. $\sqrt{5x-4}=x$
5. $\sqrt{x-2}=4-x$
6. $2+\sqrt{4x-3}=x$
7. $\sqrt{x}+\sqrt{x+5}=5$
8. $\sqrt{2x+1}-\sqrt{x}=1$

2. Solve each of the following equations:

1. $\sqrt{x+3} = 4$

2. $\sqrt{2x+3} - x = 0$

3. $\sqrt{x-2} = x - 4$

4. $5 - \sqrt{x+7} = x$

5. $\sqrt{3x+1} + 3 = x$

6. $\sqrt{3-2x} - 2x = 9$

7. $2x = \sqrt{13+2x} - 7$

8. $7 = \sqrt{5-2x} - 2x$

9. $\sqrt{10+2x} - 8 = 2x$

10. $\sqrt{3x-2} - \sqrt{2x+4} = 0$

11. $\sqrt{x} + \sqrt{6x+1} = 3$

12. $\sqrt{6x+1} - \sqrt{x} = 3$

13. $\sqrt{5x-1} - x = 1$

14. $\sqrt{x} + \sqrt{5-x} = 3$

15. $\sqrt{2x} + \sqrt{11-x} = 5$

16. $\sqrt{5-x} - \sqrt{x-1} = 2$

Topic 3: Equations Involving Absolute Values

Competencies

- Apply the definition of absolute value
- Solve equations involving absolute value

Suggested ways of teaching this topic: Teacher assisted jigsaw groups

Starter Activities

Obviously, the teacher shall start from the definition of absolute value. As students were familiar with it in the lower grades, the teacher shall ask students to state the definition of absolute values.

Expected Answer: $|x| = x$ if $x>0$, $|x| = -x$ if $x<0$ and $|x| = 0$ if $x = 0$.

Alternatively students might explain as:

When you take the absolute value of a number, you always end up with a positive number (or zero). Whether the input was positive or negative (or zero), the output is always positive (or zero). For instance, | 3 | = 3, and | –3 | = 3 also.

If consensus is reached on the understanding of the definition the teacher may start with something simple like the following to be done in pairs or in small groups:

Example: Solve the following equations

a) $| x | = 6$
b) $| x + 2 | = 7$
c) $| x - 1 | = -3$
d) $| 3 - x | = 5$

Solutions:

a) If you have $x = -6$, then " $-x$ " indicates "the opposite of x", or, in this case, $-(-6) = +6$, a positive number. The minus sign in " $-x$ " just indicates that you are changing the sign on x. It does *not* indicate a negative number comes out of the absolute value. This distinction can be crucial.

   Thus, the solution is $\boldsymbol{x = -6, 6}$

b) Solve $| x + 2 | = 7$

   To clear the absolute-value bars, we must split the equation into its possible two cases, one case for each sign:

$(x + 2) = 7$ or $-(x + 2) = 7$

$x + 2 = 7$ or $-x - 2 = 7$

$x = 5$ or $-9 = x$

Thus, the solution is ***x* = –9, 5.**

c) $|x - 1| = -3$ is meaningless! As from the definition an absolute value can't be negative.

Thus, no solution is available.

d) $|3 - x| = 5$ here, we shall split the equation into its two possible two cases

$(3 - x) = 5$ or $-(3 - x) = 5$

$3 - x = 5$ or $-3 + x = 5$

$x = -2$ or $x = 8$

Thus, the solution is ***x* = –2, 8.**

Lesson Notes

The teacher can give equations of the following type to small groups where each group takes one question. At the end of the group work, let group leaders' move to other groups to share their solutions and learn the others' solutions.

Example: Solve the following equations

a) $|2x + 3| = 2$
b) $|3 - 2x| = 1$
c) $|2(x - 3)| = 4$
d) $|2x - 3| - 4 = 3$

Solutions:

a) $|2x + 3| = 2$; split the equation into its two possible cases

$2x + 3 = 2$ or $-(2x + 3) = 2$

$2x + 3 = 2$ or $-2x - 3 = 2$

$2x = -1$ or $-2x = 5$

$x = \frac{-1}{2}$ or $\frac{-5}{2}$

b) $| 3 - 2x | = 1$; split the equation into its two possible cases

$3 - 2x = 1$ or $-(3 - 2x) = 1$

$3 - 2x = 1$ or $-3 + 2x = 1$

$-2x = -2$ or $2x = 4$

$x = 1$ or $x = 2$

c) $| 2(x - 3) | = 4$; split the equation into its two possible cases

$2(x - 3) = 4$ or $-2(x - 3) = 4$

$2x - 6 = 4$ or $-2x + 6 = 4$

$2x = 10$ or $-2x = -2$

$x = 5$ or $x = 1$

d) $| 2x - 3 | - 4 = 3$

First, we will isolate the absolute-value part; that is, we will get the absolute-value expression by itself on one side of the "equals" sign, with everything else on the other side:

$$| 2x - 3 | = 7$$

Now we will clear the absolute-value bars by splitting the equation into its two cases, one for each sign:

$(2x - 3) = 7$ or $-(2x - 3) = 7$
$2x - 3 = 7$ or $-2x + 3 = 7$
$2x = 10$ or $-2x = 4$
$x = 5$ or $x = -2$

Concluding Activities

The teacher shall make sure that the students got the idea of solving equations involving absolute value by asking them to tell him/her the general format of solving these types of equations. The students are expected to give the following summary:

Whatever the value of x might be, taking the absolute value of x makes it positive. Since x might have been positive and might have been negative, we have to acknowledge this fact when we take the absolute-value bars off, and you do this by splitting the equation into two cases. If the value of x was positive to start with, then you can bring that value out of the absolute-value bars without changing its sign. But x might also have been negative, in which case we would have to change the sign on x for the absolute value to come out positive.

To be able to remove the absolute-value bars, you have to isolate the absolute value onto one side, and then split the equation into the two possible cases. Then, start solving the two linear equations obtained from the two cases.

That is, $|x| = x$ if $x > 0$, $|x| = - x$ if $x < 0$ and $|x| = 0$ if $x = 0$.

Practice Exercises

Solve the following equations

a) $| 4x - 1 | = 0$
b) $| 2x + 3 | = - 1$
c) $| 5x + 3 | = 1$
d) $| -5x - 1 | - 4 = 2$
e) $| -2x + 3 | - 2 = - 1$
f) $| - 3 - 2x | = 2$
g) $| - 1 - 5x | + 4 = 2$
h) $| -2(x + 3) | + 2 = 3$

Competencies

- Identify the terms factor, trinomial,
- Factorize trinomials

Suggested ways of teaching this topic: Teacher's explanation with guided practice

Starter Activities

The process of factoring is essentially the opposite of the FOIL Method, which is a process used to multiply two binomials. At the beginning, the teacher shall make sure the students understand the FOIL Method.

The teacher could ask students to examine the following expression, in pairs or small groups, which consists of one binomial in parentheses multiplying another binomial in parentheses.

$(2k + 7)(3k - 10)$

Let students use the FOIL Method to simplify the expression.

Expected answer is the work below and a final result which is a trinomial.

$6k^2 - 20k + 21k - 70$
$6k^2 + k - 70$

The teacher shall make students notice that a trinomial consists of three terms.

Lesson Notes

Then the teacher should ask students "how can we go from the trinomial to factors or factor a trinomial like this into two binomials?"

The first problem could be of the following type:

$\mathbf{m^2 + 10m + 16}$

Before attempting to factor any more, there are a few simple questions you can ask to make sure that the expression is factorable as a trinomial.

Question	Answer and Reason
After the GCF has already been factored out, are all variables in the first (m^2) and last (16) terms to an even power?	*Yes. (m^2 is the only variable in the first and last terms, and its exponent, 2, is an even number.)*
Are all of the middle term's (10m) exponents to half of the power of one of the exponents on the outside terms?	*Yes. (In 10m, the middle term, m is the only variable. Its exponent is not shown and therefore is 1 which is half of 2, the exponent of m in the first term.)*

The next step is to write sets of open parentheses, side by side.

(   )(   )

Handling Variables

On the left side of each set of parentheses, write all of the variables from the first term of the trinomial with half their exponents. The first term is m^2, thus after the exponent is cut in half, m is placed inside each set of parentheses.

(m   )(m   )

Since all of the terms in the original set of parentheses are positive, two plus signs are placed below.

(m +   )(m +   )

Handling Coefficients

Now identify the coefficient of the first and last terms in the original set of parentheses, $m^2 + 10m + 16$. The coefficients are 1 and 16. Now write all pairs of factors of 1 in a vertical column and then write all pairs of factors of 16 in another vertical column.

Factors of 1	Factors of 16
1 * 1	1 * 16
	2 * 8
	4 * 4

Choosing Factor Pairs

Now we must choose a pair of factors of 1 and a pair of factors of 16 to insert into the pairs of parentheses. But how is this done? For the number 1, there is only 1 pair of factors, eliminating any choice. Thus, the 1s can be inserted on the left side of each set of parentheses.

(1m +   )(1m +   )

Finally, a pair of factors of 16 must be chosen. This is a matter of trial and error. To make sure all pairs are considered, start using factors at the top of the list then check each pair below it, in order.

We start by choosing the first pair, 1 * 16. We place the factors on the right side of each set of parentheses.

$(1m + 1)(1m + 16)$

Now test whether this is the correct pair. Multiply the expression using the FOIL Method:

$(1m^2 + 16m + m + 16)$
$(1m^2 + 17m + 16)$

Note that the resulting trinomial is not the same as the one we started with, so this is the incorrect pair of factors of 16. Try the next pair:

$(m + 2)(m + 8)$
$(m^2 + 8m + 2m + 16)$
$(m^2 + 10m + 16)$

Since the result here is equivalent to the original, the correct pair of factors of 16 has been chosen. Additionally, the answer to the problem is: $(m + 2)(m + 8)$

Factoring a Trinomial with a Negative Sign

Example 1: Examine this expression.

$s^2 - 5s + 6$

Only one of the three terms is negative. As a result, a minus sign should not be factored out as in the previous example. This expression will be factored much like the expression on the previous page, but we will need to work with the minus sign as we build two sets of parentheses.

To start factoring, first write out two empty sets of parentheses.

( )( )

Handling Variables

Now on the left of each set of parentheses, write all of the variables from the first term with half the exponent. The first term is s^2, after dividing its exponent by two; we place the variable s on the left of each set of parentheses.

(s)(s)

The last term in the trinomial does not have any variables, so we do not need to carry any variables into the parentheses as a result.

Handling Coefficients

Now identify the coefficient of the first and last terms in the original set of parentheses, $s^2 - 5s + 6$. The coefficients are 1 and 6. Now write all pairs of factors of 1 in a vertical column and write all pairs of factors of 6 in another vertical column.

Since this problem involves a negative term, we also include the negative factors of 6.

Factors of 1	**Factors of 6**
1 * 1	1 * 6
	-1 * -6
	2 * 3
	-2 * -3

Choosing Factor Pairs

Now we must choose a pair of factors of 1 and a pair of factors of 16 to insert into the pairs of parentheses. For the number 1, there is only one pair of factors, eliminating any choice. Thus the 1s can be inserted on the left side of each set of parentheses.

(1s)(1s)

Now a pair of factors of 6 must be chosen. This is a matter of trial and error. To make sure all pairs are considered, check each pair of factors starting with the pair on the top of the list and working toward the bottom of the list.

A summary of the trial and error process using the FOIL Method is below.

$(s + 1)(s + 6) = s^2 + 6s + s + 6 = s^2 + 7s + 6$

$(s + -1)(s - 6) = s^2 - 6s - s + 6 = s^2 -7s + 6$

$(s + 2)(s + 3) = s^2 + 3s + 2s + 6 = s^2 + 5s + 6$

$(s - 2)(s - 3) = s^2 - 3s - 2s + 6 = s^2 - 5s + 6$

The expression $s^2 - 5s + 6$ is equivalent to the original expression, therefore (s - 2)(s - 3) is the final answer.

An Example with Two Negative Signs

The problem below is similar to the last problem, but it has two negative signs in the expression.

Example 2: Factor $p^2 - 20p - 21$

First, write out two sets of empty parentheses.

( )( )

Write all of the variables from the first term with half the exponent in the front of each set of parentheses. Again, since there are no variables in the last term, nothing is written on the right side of each set of parentheses.

(p)(p)

Write out the factors of the coefficients of the first and last terms. The previous problems have led to the observation that the first term's coefficient of 1 results in 1 * 1 as the pair of factors chosen.

(p)(p)

Now the pairs of factors of 21 (both positive and negative) are computed.

Factors of -21
1 * -21
-1 * 21
3 * -7
-3 * 7

Use trial and error to find which pair of factors to use.

$(p + 1)(p - 21) = p^2 - 21p + 1p - 21 = p^2 - 20p - 21$

The expression $p^2 - 20p - 21$ is equivalent to the original expression. Therefore, $(p + 1)(p - 21)$ is the answer.

Sum and Product Method

Example 3: Factor $6c^2 + c - 12$

The expression is a trinomial, like the expressions which we factored on previous pages.

There is a small variation in this problem; however, the first term has a coefficient that is not 1.

This expression will be factored much like the others, the main difference is that we will need to find the correct pair of factors for the first term's coefficient (6) in addition to the correct pair of factors for the last term's coefficient (-12).

Thus, try to find two numbers whose product is -72 and whose sum is 1.

To do this, we need to write possible factors of -72.

Factors of -72
1 * -72
-1 * 72
2 * -36
-2 * 36

3 * -24
-3 * 24

4 * -18
-4 * 18
6 * -12
-6 * 12

8 * -9
-8 * 9

The last pair satisfies our condition, sum = 1 and product = -72.

Thus, the middle term of the trinomial can be written as -8c +9c while the leading term as (2c)(3c) and the last term as (3)(4).

We can now have $6c^2 + c - 12 =$ [(2c)(3c)– 8c]+[9c –(3)(4)]

Taking 2c a common term the first sum and 3 from the second we can have:

2c (3c – 4) + 3(3c – 4).

This is same as distributing (3c – 4) in to (2c + 3).

Thus, the factors are (3c – 4) and (2c + 3).

Example 4: Factorize $-5x^2 + 4x + 1$

Solution: Start by taking out the minus sign from the leading term.

We get:- $(5x^2 - 4x - 1)$

Then, leaving the minus sign as it is we can factor the trinomial in the bracket.

Here, we need to find two numbers whose product is (5)(-1)= -5 and whose sum is –4.

The teacher may ask: “Which do you think are those numbers?” ‘why?” to make sure that students know what they are doing.

Expected Answer: -5 and 1.

Then we can write the given trinomial as follows:

$-5x^2 + 4x + 1 = -(5x^2 - 4x - 1) = - (5x.x -5x + x - 1)$

$= - [5x(x - 1) + 1(x-1)]$

$= -(5x + 1)(x-1)$

Thus, the factors -(5x + 1) and (x-1).

Remember students should be reminded about the trinomials that can't be factored using sum and product rule. Some examples are $2x^2 - 4x + 5$; $x^2 - x + 1$;

Concluding Activities

The teacher has to make sure that the students have understood the process of factoring. To factor trinomials of the form $ax^2 + bx + c$, we need to find two numbers whose sum is **b** and whose product is **a•c**. If we can find the pair, say p and q, we can write the middle term as a sum of px+qx. Then, what remains is taking out common and writing the factors.

Practice Exercises

1. Find the products

   a) $(2x + 7)(5x + 1)$
   b) $(3x - 1)(2x+3)$
   c) $(2 - 5x)(x - 1)$
   d) $-(x - 3)(3x + 1)$

2. Factor the following trinomials

   a) $x^2 - 9$
   b) $3x^2 - 1$
   c) $x^2 - 6x + 9$
   d) $x^2 + 4x - 5$
   e) $3x^2 - 4x + 3$
   f) $8 - 2c - c^2$
   g) $-5m - m^2 - 4$

Topic 5: Solving Quadratic Equations using Completing the Square

Competencies

- Perform completing square
- Solving quadratic equations using completing the square

Suggested ways of teaching this topic: Jigsaw in small groups

Starter Activities

The teacher might start with some quadratic equations which are fairly simple to solve like those of the form "something-with-*x* squared equals some number", and then with trinomials. An example would be:

$(x – 4)^2 = 5$
$x – 4 = \pm\sqrt{5}$
$x = 4 \pm \sqrt{5}$
$x = 4 – \sqrt{5}$ and $x = 4 + \sqrt{5}$

Unfortunately, most quadratic equations don't come neatly squared like this. Thus, students shall learn how to use the technique of "completing the square" to rearrange the quadratic into the neat "(squared part) equals (a number)" format demonstrated above.

The teacher can demonstrate the steps using one of the following examples. Then, the teacher can prepare similar questions and give each group one question. Let students work in their groups for some time and then disperse the home group and form other groups where each student can share other groups' solutions.

Lesson Notes

Example 1:

Solve the equation: $4x^2 – 2x – 5=0$.

Solution:

This is the original problem.	$4x^2 - 2x - 5 = 0$
Move the loose number over to the other side.	$4x^2 - 2x = 5$
Divide through by whatever is multiplied on the squared term. Take half of the coefficient (don't forget the sign!) of the *x*-term, and square it. Add this square to both sides of the equation. Convert the left-hand side to squared form, and simplify the right-hand side. (This is where you use that sign that you kept track of earlier. You plug it into the middle of the parenthetical part.)	$x^2 - \frac{1}{2}x = \frac{5}{4}$ $-\frac{1}{4} \to \frac{1}{16}$ $x^2 - \frac{1}{2}x + \frac{1}{16} = \frac{5}{4} + \frac{1}{16}$ $\left(x - \frac{1}{4}\right)^2 = \frac{21}{16}$
Taking the square root both sides, remembering the "±" on the right-hand side. Simplify as necessary.	$x - \frac{1}{4} = \pm\sqrt{\frac{21}{16}} = \pm\frac{\sqrt{21}}{4}$
Solve for "*x* =".	$x = \frac{1}{4} \pm \frac{\sqrt{21}}{4}$
Remember that the "±" means that you have two values for *x*.	$x = \frac{1}{4} - \frac{\sqrt{21}}{4}$ and $x = \frac{1}{4} + \frac{\sqrt{21}}{4}$

Students should be reminded that when they complete the square, they should make sure that they are careful with the sign on the *x*-term when they multiply by one-half. If they lose that sign, they can get the wrong answer in the end, because they might forget what goes inside the parentheses. Also, careful not to be sloppy and wait to do the plus/minus sign until the very end.

Example 2: Solve x2 + 6x – 7 = 0 by completing the square.

Do the same procedure as above, in exactly the same order. (Study tip: Always working these problems in exactly the same way will help you remember the steps when you're taking your tests.)

This is the original equation.	$x^2 + 6x - 7 = 0$
Move the loose number over to the other side.	$x^2 + 6x = 7$
Take half of the *x*-term (that is, divide it by two) (and don't forget the sign!), and square it. Add this square to both sides of the equation.	$x^2 + 6x \qquad = 7$
Convert the left-hand side to squared form. Simplify the right-hand	$(x + 3)^2 = 16$

side.	
Square-root both sides. Remember to do "±" on the right-hand side.	$x + 3 = \pm 4$
Solve for "x =". Remember that the "±" gives you two solutions. Simplify as necessary.	$x = -3 \pm 4$ $= -3 - 4, -3 + 4$ $= -7, +1$

Remind students if they are not consistent with remembering to put the plus/minus in as soon as they take square root both sides, then this is an example of the type of exercise where they will get themselves in trouble. They will write your answer as "$x = -3 + 4 = 1$", and have no idea how they got "$x = -7$", because they won't have a square root symbol "reminding" them that they "meant" to put the plus/minus in. That is, if they miss, these *easier* problems will embarrass themselves!

Example 3: Solve$x^2 + 6x - 10 = 0$.

Apply the same procedure as on the previous page:

This is the original equation.	$x^2 + 6x - 10 = 0$
Move the loose number over to the other side.	$x^2 + 6x = 10$
Take half of the coefficient on the x-term (that is, divide it by two, and keeping the sign), and square it. Add this squares value to both sides of the equation.	$x^2 + 6x + 9 = 10 + 9$
Convert the left-hand side to squared form. Simplify the right-hand side.	$(x + 3)^2 = 19$ $x + 3 = \pm\sqrt{19}$
Square root both sides. Remember to put the "±" on the right-hand side.	$x = \pm\sqrt{19}$
Solve for "x =", and simplify as necessary.	$x = -3 \pm \sqrt{19}$

Quadratic Formula

After students' successful understanding of the process of completing the square, the next discussion would be on performing completing the square for the generic quadratic equation written as $ax^2 + bx + c = 0$. Here, the teacher can ask students try to do it by themselves in their groups. With the assistance of the teacher they might be able to come up with the quadratic formula.

Though it is understood that there is a difficulty teacher might face (showing students how the Formula was invented, and thereby giving an example of the usefulness of symbolic manipulation), the computations involved are often a bit beyond the average student at this point. But, if explained well any student could get it easily. Here is what the teacher is looking for:

Deriving the Quadratic Formula starts by solving the general quadratic equation $ax^2 + bx + c = 0$, a 0. This is the original equation.	$ax^2 + bx + c = 0$
Move the loose number to the other side.	$ax^2 + bx = -c$
Divide through by whatever is multiplied on the squared term. Take half of the *x*-term, and square it. Add the squared term to both sides.	$x^2 + \frac{b}{a}x = -\frac{c}{a}$ $\frac{b}{2a} \rightarrow \frac{b^2}{4a^2}$ $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$
Simplify on the right-hand side; in this case, simplify by converting to a common denominator.	$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2}$
Convert the left-hand side to square form (and do a bit more simplifying on the right).	$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$
Square root both sides, remembering to put the "±" on the right.	$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$
Solve for "*x* =", and simplify as necessary.	$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Conclusion

Whether students are working symbolically (as in the last example) or numerically (which is the norm), the key to solving by completing the square is to practice, practice and practice. By so doing, the process will become a bit more "automatic", and they will remember the steps when they are asked any time. Thus, teacher should give practice exercises.

Practice Exercises

1. Factorize the following expressions:

a) $57x^2 – 21 + 19 – 7x^2$
b) $x^4 – 16$
c) $16 – 40x^2 + 9 + 4x^2$
d) $3x^4 – 4x^3$h) $(x – 1)^2 – 4$
e) $5(x – 8)^2 + 8 – 133$
f) $24x^2 – 36x + 30$
g) $1 – 9(x + 3)^2$

2. Solve for the unknown in each of the following:

a) $5x (3x - 7) = 0$
b) $12x^2 – 3 = 0$

c) $4x^2 + 6 = 11x$
d) $(2x - 3)^2 – 4 = 0$

3. Solve for x in each of the following
   a) $3x^2 – 11 = 14 – 13x^2$
   b) $(x – 1)^2 – 4 = 0$
   c) $3x^2 – 14x = x^2 – 13x$
   d) $3x^2 = 10 – 13x$
   e) $(x + 8)^2 = 25$
   f) $x^2 + 2x = 8x$
   g) $5x^2 + 3 = 21 + 3x^2$
   h) $2x^2 + x\ 14 – 2x^2$
4. Use completing the square to solve the following quadratic equations:
   a) $2x^2 – 5x + 3 = 0$
   b) $9x^2 + x + 1 = 0$
   c) $4x^2 + 4x + 1 = 0$
   d) $x^2 – 5x – 3 = 0$

5. Which one of the following is an equivalent form of the quadratic expression $3x^2 + 12x + 1$?

A. $3(x+2)^2 + 13$ *C.* $3(x+2)^2 - 11$
B. $3(x+2)^2 - 1$ *D.* $3(x+2)^2 - 13$

!

- State Viete's theorem
- Apply Viete's theorem to solve problems

Suggested ways of teaching this topic: Presentation of the teacher followed by individual consolidation and practice activities

Starter Activities

Before coming to the discussion on Viete's theorem and its application, the students shall understand the nature of the roots of a quadratic equation. The teacher might start from "A Question and Answer Session" where students can be asked to solve the following quadratic equation and find out the relationship between the roots and the coefficients of the equation. The process goes as follows:

Example: Solve each of the following equations by using the quadratic formula:

a) $2x^2 - 3x - 2 = 0$
b) $4x^2 - 4x + 1 = 0$
c) $3x^2 - 2x + 1 = 0$

Solution:

a) $2x^2 - 3x - 2 = 0$

$a = 2, b = -3, c = -2$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-2)}}{2(2)}$$

$$x = \frac{3 \pm \sqrt{9 + 16}}{4}$$

$$x = \frac{3+5}{4} \text{ or}$$

$$x = \frac{3-5}{4}$$

$$x = 2 \quad or \quad x = -\frac{1}{2}$$

b) $4x^2 - 4x + 1 = 0$

$a = 4, b = -4, c = 1$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(4)(1)}}{2(4)}$$

$$x = \frac{4 \pm \sqrt{0}}{2(4)} = \frac{4}{8} = \frac{1}{2}$$

c) $3x^2 - 2x + 1 = 0$

$a = 3, b = -2, c = 1$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - (4)(3)(1)}}{2(3)}$$

$$x = \frac{-2 \pm \sqrt{4 - 12}}{6}$$

$$x = \frac{-2 \pm \sqrt{-8}}{6}$$

Since $\sqrt{-a}$, where $a > 0$, is undefined; this equation has **no solution.**

At this stage, the teacher shall ask students to observe the value of $b^2 - 4ac$ (the expression inside the square root)for each of the three quadratics. The expected results are:

For a); $b^2 - 4ac = 25$, which is greater than 0

For b); $b^2 - 4ac = 0$

For c); $b^2 - 4ac = -8$, which is less than zero.

The next is asking an important question: **“What do you observe from the sign of $b^2 - 4ac$ and the number solutions of these equations?”**

Expected Answer

- From the examples solved above, it is clear that the expression $b^2 - 4ac$ is the key to determine the nature of roots of a given quadratic equation.
- The expression $b^2 - 4ac$ is called the discriminate and its value will enable us to make predictions regarding the types of roots that we will obtain.

Lesson Notes

Let $D = b^2 - 4ac$.

1. If $D > 0$, then the equation ha 2 distinct solution
2. If $D = 0$, then the equation has exactly 1 solution
3. If $D < 0$, then the equation has No solution

Example:

Find the discriminate of the following equations and determine their nature of roots, without actually solving any of them:

a) $x^2 + 2x - 5 = 0$
b) $2x^2 - x + 1 = 0$
c) $x^2 + 5x - 6 = 0$
d) $x^2 + 4x + 4 = 0$

Solution:

a) $x^2 + 2x – 5 = 0$

$a= 1, b= 2, c = -5$

$D = b^2 – 4ac = 2^2 – (4)(1)(-5) = 24$

$D > 0$ 2 distinct Solution

b) $2x^2 – x + 1 = 0$

$a = 2, b = -1, c= 1$

$D=b^2–4ac=(-1)^2 (0) - (4)(2)(1) = -7$

$D < 0$ No solution

c) $x^2 + 5x – 6 = 0$

$a = 1, b = 5, c = -6$

$D = b^2 – 4ac = 5^2 – 4(1)(-6) = 25 + 24 = 49$

$D > 0$ two distinct solutions

d) $x^2 + 4x +4 = 0$

$a = 1, b = 4, c = 4$

$D = b^2 – 4ac = 4^2 – 4(1)(4)$

$= 16 – 16 = 0$

$D = 0$ Exactly 1 solution

Applications of the Nature of Roots

Example 1:

For which values of k will the equation $2x^2+kx + 1 = 0$ have equal roots?

Solution: $a = 2, b = k$ and $c = 1$

$D = b^2 – 4ac = (k)^2 -4(2)(1)$

$= k^2 – 8$

But D = 0 (equal roots)

$k^2 – 8 = 0$

$k = \pm\sqrt{8}$

Example 2:

Find the value of k for which the quadratic equation$x^2 - 8x + k =0$ will have exactly one solution.

Solution: $a = 1, b= -8, c = k$

$D = b^2 – 4ac = (-8)^2 – 4(1)(k) = 64 – 4k$

But D = 0 (which means one solution)

$64 - 4k = 0$

$64 = 4k$

$k = 16$

Example 3:

Find the value(s) of k for which the quadratic equation $3x^2 - 2kx + 3 = 0$ will have exactly one solution.

Solution:

$3x^2 - 2kx + 3 = 0$

$a = 3, b = -2k, c = 3$

$D = b^2 - 4ac = (-2k)^2 - 4(3)(3)$

$= 4k^2 - 36$

But $D = 0$ $\quad 4k^2 = 36$

$k = \pm 3$

Now, before stating the theorem, the teacher could take one of the quadratic equations with two roots and ask students to

1. add the roots, then compare the sum with the value of $\frac{-b}{a}$ for that equation
2. multiply the roots , then compare the product with the value of $\frac{c}{a}$ for that equation

If $x^2 + 5x - 6 = 0$ were taken the roots are $r_1 = -6$ and $r_2 = 1$.

Thus, $r_1 + r_2 = -6 + 1 = -5 = \frac{-b}{a}$ and $r_1 \times r_2 = -6 \times 1 = -6 = \frac{c}{a}$

This is time the theorem that explains the existing special relationship between the roots of a quadratic equation and its coefficients can be stated.

Viete's Theorem

Theorem: If the root of $ax^2 + bx + c = 0$, a 0 are r_1 and r_2, then

a) $r_1 + r_2 = \frac{-b}{a}$

b) $r_1 r_2 = \frac{c}{a}$

Example 4: Check Viete's theorem using $2x^2 + 3x - 5 = 0$

a = 2, b = 3, c = -5

*Checking Viete's Theorem***:**

Sum: $r_1 + r_2 = 1 + \left(-\frac{5}{2}\right) = -\frac{3}{2} \rightarrow \frac{-b}{a}$

Product: $r_1 r_2 = 1 \times \left(-\frac{5}{2}\right) = -\frac{5}{2} \rightarrow \frac{c}{a}$

Example 5: Without computing the roots α and β of the equation $3x^2 + 2x + 6 = 0$ find

1. $\alpha + \beta$
2. $\alpha\beta$
3. $\frac{1}{\alpha} + \frac{1}{\beta}$

Solution:

From $3x^2 + 2x + 6 = 0$, a = 3, b = 2, c = 6

1) $\alpha + \beta = \frac{-b}{a} = \frac{-2}{3}$

2) $\alpha\beta = \frac{c}{a} = \frac{6}{3} = 2$

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta}$$

$$\frac{-\frac{2}{3}}{2} = \frac{-1}{3}$$

Example 6*:* Without computing the roots α and β of the equation $2x^2 - 5x - 12 = 0$, find

a) $\frac{1}{\alpha} + \frac{1}{\beta}$

b) $\alpha^2 + \beta^2$

Solutions:

$2x^2 - 5x - 12 = 0 \qquad a = 2, b = -5 , c = -12$

a) $$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta} = \frac{\frac{5}{2}}{-6} = \frac{-5}{12}$$

b) $$(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$$

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$= \frac{5}{2}^{\,2} - 2(-6)$$

$$= \frac{25}{4} + 12$$

$$= 18\frac{1}{4}$$

Practice Exercises

1. Find the discriminate for each of the listed equations and determine their **nature of roots**, without actually solving any of the equations:

   a) $2x^2 - 5x + 3 = 0$
   b) $9x^2 + x + 1 = 0$
   c) $4x^2 + 4x + 1 = 0$
   d) $x^2 - 5x - 3 = 0$

2. How many solutions does each equation have?

   a) $3x^2 - x + 2 = 0$
   b) $2x - x^2 + 15 = 0$
   c) $1 - 2x + x^2 = 0$

3. If one of the roots and the product of the roots of the equation $6x^2 + Mx + N = 0$ respectively are ½ and -2/3 , what are the values of M and N?
4. For what value of k will $(k+3)x^2 + (3k+1)x + 1 = 0$ have only one root?
5. A quadratic equation has two unequal roots. If the difference between them is 1 and the difference of their squares is 2, what will be the equation?

For the following questions choose the best answer

1. Which one of the following equations has two distinct roots?

$A.\ \ 6x^2 - x - 1 = 0 \qquad C.\ \ 4x + 1 = 0$

$B.\ \ 2x^2 - 5x + 4 = 0 \qquad D.\ \ .x^2 + 2x + 1 = 0$

2. Which one gives the sum of the roots of the quadratic equation $\sqrt{2}x^2 - \sqrt{3}x + 11 = 0$?

$A.\ \ \frac{-11\sqrt{2}}{2} \quad B.\ -\sqrt{6}/2 \quad C.\ \ 11\sqrt{2}/2 \quad D.\ \ \sqrt{6}/2$

3. If -2 and 3 are the roots of $x^2 + Mx - N = 0$, then what are the values of M and N respectively?

$A.\ \ 1\ and -6 \qquad C.\ \ 6\, and -1$

$B.\ \ -6\, and\ 1 \qquad D.\ \ -1\ and\ 6$

5. If one root of $x^2 + K = 0$ is 5, for some constant k, then the other solution is:

   A. -25
   B. 25
   C. -5
   D. 5

6. If k is a constant number and $x^2 + 2x + k = 0$ has <u>at most</u> one real root, then what is the minimum value of k?

$A.\ \ 0 \quad B.\ 1 \quad C.\ -1 \quad D.\ \ there\ is\ no\ such\ k.$

7. The set of values of k for which $kx^2 - kx + 2 = 0$ has only one root is:

$A.\ \{0,8\} \quad B.\ \{8\} \quad C.\ \phi \quad D.\ \{-1,2\}$

8. If one of the roots of the equation $x^2 - 8x + k = 0$ exceeds the other by 4 then k is equal to:

   A. 12
   B. 15
   C. 16
   D. 6

" # $

Competencies

- Describe the terminology used with sets and Venn diagrams.
- Determine the placement of an element in a Venn diagram.
- Apply the notion of Venn diagrams to solve problems related to sets.

Suggested ways of teaching this topic: teacher explanation, guided practice and individual practice

Starter Activities

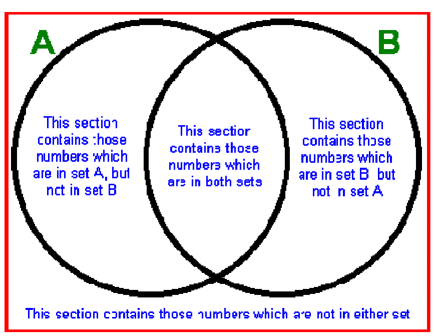
The teacher may lead the students in a short discussion about Venn Diagrams. This topic is where we introduce the ideas of sets and Venn diagrams. A *set* is a list of objects in no particular order; they could be numbers, letters or even words. A *Venn diagram* is a way of representing sets visually.

Teacher explanation and input

To explain, the teacher could start with an example where we use whole numbers from 1 to 10.
We will define two sets taken from this group of numbers:

Set A = the odd numbers in the group = { 1 , 3 , 5 , 7 , 9 }
Set B = the numbers which are 6 or more in the group = { 6 , 7 , 8 , 9 , 10 }

Some numbers from our original group appear in both of these sets. Some only appear in one of the sets.
Some of the original numbers don't appear in either of the two sets. We can represent these facts using a Venn diagram.



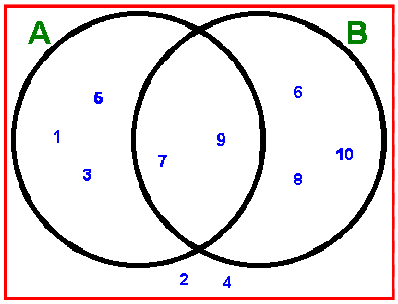
The two large circles represent the two sets.

The numbers which appear in *both* sets are 7 and 9. These will go in the central section, because this is part of *both* circles.

The numbers 1, 3 and 5 still need to be put in Set A, but not in Set B, so these go in the left section of the diagram.

Similarly, the numbers 6, 8 and 10 are in Set B, but not in Set A, so these will go in the right section of the diagram.

The numbers 2 and 4 are not in either set, so will go outside the two circles.



The final Venn diagram looks like this:

We can see that all ten original numbers appear in the diagram.

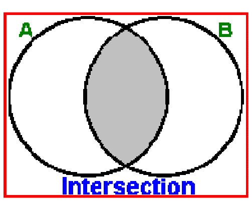
The numbers in the left circle are Set A
{ 1 , 3 , 5 , 7 , 9 }

The numbers in the right circle are Set B
{ 6 , 7 , 8 , 9 , 10 }

Guided Practice

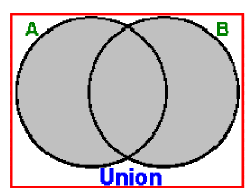
After students understood the basic ideas, the rest of the lesson will be student's practice filling in Venn diagrams and using them.

Lesson Notes



The **intersection** of sets A and B is those elements which are in *set A*and*set B*. A diagram showing the intersection of A and B is on the left.

The **union** of sets A and B is those elements which are in *set A*or*set B*or*both*. A diagram showing the union of A and B is on the right.



Setting: Ten Best Friends

The teacher could take a set made up of ten best friends listing them like:

{Alemu, Balcha, Kassa, Demis, Eshet, Fire, Genet, Hadush, Imam, Jasmin}

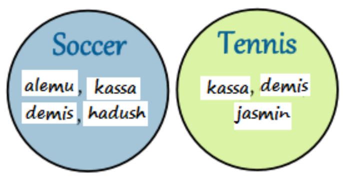
Each friend is an "element" (or "member") of the set

Now let's say that alemu, kassa, demis and hadush play **Soccer**and kassa, demis and jasmin play **Tennis**. That is,

"Soccer" = {Alemu, Kassa, Demis, Hadush}

Tennis = {Kassa, Demis, Jasmin}

You could put their names in two separate circles:



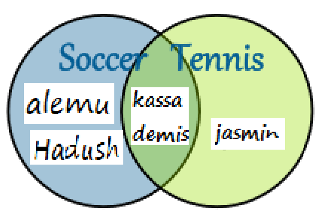
Union

You can now list your friends that play **Soccer OR Tennis**. Not everyone is in that set. Only your friends that play Soccer or Tennis.

This is called a "Union" of sets and has the special symbol :

Soccer Tennis = {Alemu, Kassa, Demis, Hadush, Jasmin}

We can also put it in a "Venn Diagram":



A Venn diagram is better because it shows lots of information easily.

- Do you see that Alemu, Kassa, Demis and Hadush are in the "Soccer" set?
- And that Kassa, Demis and Jasmin are in the "Tennis" set?
- And here is the clever thing: **Kassa and Demis are in BOTH sets!**

Intersection

"Intersection" is those who are in BOTH sets.

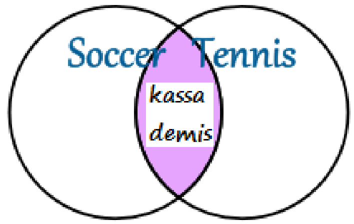
In this we mean those who **they play both Soccer AND Tennis** ... which is Kassa and Demis.

The special symbol for Intersection is an upside down "U" like this:

And this is how we write it down:

Soccer Tennis = {Kassa, Demis}

In a Venn diagram, it looks as follows:



Difference

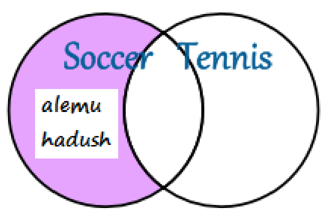
You can also "subtract" one set from another.

For example, taking Soccer and subtracting Tennis means people that **play Soccer but NOT Tennis** ... which is Alemu and Hadush.

And this is how we write it down:

Soccer **-** Tennis = {Alemu, Hadush}

In a Venn diagram, difference of two sets can be seen as follows:



Summary So Far

- is Union: is in either set
- is Intersection: must be in both sets
- **-** or \ is Difference: in one set but not the other

Three Sets

You can also use Venn Diagrams for 3 sets.

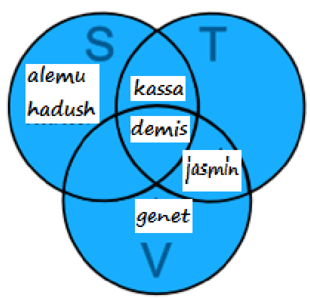
Let us say the third set is "Volleyball", which Demis, Genet and Jasmin play:

Volleyball = {Demis, Genet, Jasmin}

But let's be more "mathematical" and use a Capital Letter for each set:

- **S** means the set of Soccer players
- **T** means the set of Tennis players
- **V** means the set of Volleyball players

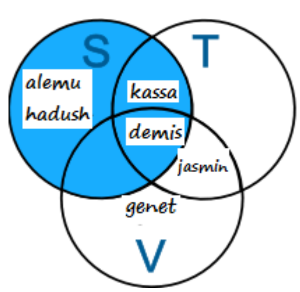
The Venn diagram Union of 3 Sets: S T V is now like this:



You can see (for example) that:

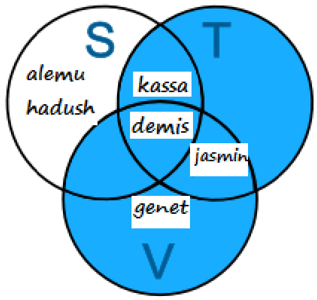
- Demis plays Soccer, Tennis **and** Volleyball
- Jasmin plays Tennis and Volleyball
- Alemu and Hadush play Soccer, but don't play Tennis or Volleyball
- no-one plays **only** Tennis

We can now have some discussions on Venn diagrams of Unions and Intersections.



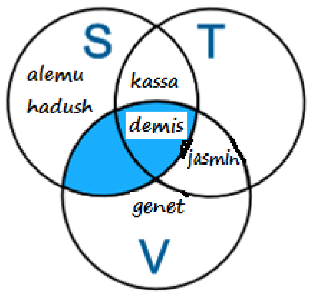
This is just the set S

S = {Alemu, Kassa, Demis, Hadush}



This is the Union of Sets T and V

T V = {Kassa, Demis, Jasmin, Genet}



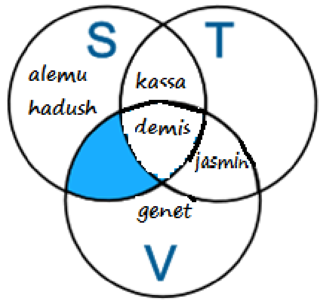
The shaded part shows the **Intersection** of Sets S and V. (only Demis is available)

Thus, S V = {Demis}

The teacher may ask students as: “how about performing the following?”

- take the **previous set** S V
- then **subtract T**:

Expected Answer



This is the Intersection of Sets S and V **minus** Set T

(S V) - T = {}

Look, there is nothing there!

That is just the "Empty Set". It is a null set, so we use the curly brackets with nothing inside: {}

Universal Set

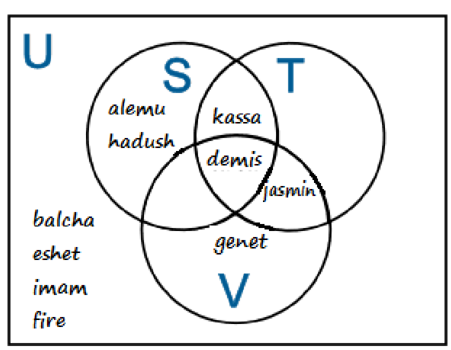
The **Universal Set** is the set that contains **Everything that we are interested in the discussion.**

Sadly, the symbol is the letter "U" ... which is easy to confuse with the for Union. Students must be reminded to be careful not to mix up the symbols.

In our case the Universal Set is our Ten Best Friends.

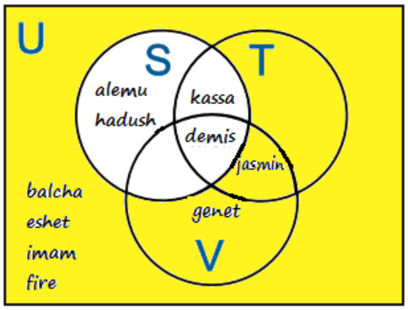
U = {Alemu, Balcha, Kassa, Demis, Eshet, Fire, Genet, Hadush, Imam, Jasmin}

We can show the Universal Set in a Venn diagram by putting a box around the whole thing:



Now you can see ALL your ten best friends, neatly sorted into what sport they play (or not!).

And then we can do interesting things like take the whole set and **subtract the ones who play Soccer**:



We write it this way:

U –S = {Balcha, Eshet, Fire, Genet, Imam, Jasmin}

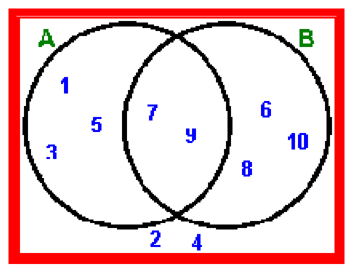
U – S means: "The Universal Set minus the Soccer Set is the Set {Balcha, Eshet, Fire, Genet, Imam, Jasmin}". In other words, U – S is: "everyone who does **not** play Soccer".

Independent practice

- Allow the students to work by themselves and to complete a worksheet, the teachershould prepare and provide one. Monitor them for questions and to be sure that the students are working
- Students may need help with some of the later questions. Not all of the Venn diagram questions are math related. Some relate to science, and some to common knowledge, in order to allow students to practice Venn diagrams more fully. Thus, teacher shall help the class to talk about what an unknown word could be--chances are good that if one student does not know what a word means, someone else in the class will.

Example 1: Given the following Venn diagram, answer each of the following questions

(a) Which numbers are in the union of A and B?
(b) Which numbers are in the intersection of A and B?



Solution:

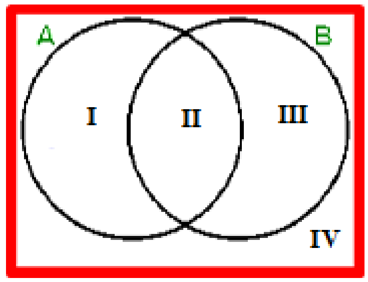
(a) $A \cup B = \{1,3,5,7,9,6,8,10\}$

(b) $A \cap B = \{7, 9\}$

Example 2: For any two non-empty sets A and B, Use Venn diagrams to show that:

a) $(A\cup B)' = A' \cap B'$ **b)** $(A \cap B)' = A' \cup B'$

Solution:



Looking at the following Venn diagram,

a) $\mathbf{A\cup B}$ can be seen on regions **I, II** and **III**
So, $\mathbf{(A\cup B)'}$ is seen in region **IV**

On the other hand, $\mathbf{A'}$ is covered by regions **III** and **IV,** while B' is covered by regions **I** and **IV.**

So, $\mathbf{A' \cap B'}$ is region **IV.** This shows that $\mathbf{(A\cup B)'}$ and $\mathbf{A' \cap B'}$ are represented by the same region, IV.

Thus, $\mathbf{(A\cup B)' = A' \cap B'}$

b) $\mathbf{A\cap B}$ is seen on region **II**
So, $\mathbf{(A\cap B)'}$ is seen in regions **I, III** and **IV**

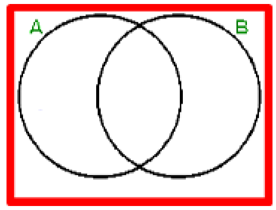
On the other hand, $\mathbf{A'}$ is covered by regions **III** and **IV,** while B' is covered by regions **I** and **IV.**

So, $\mathbf{A' \cup B'}$ is represented by regions **I, III** and **IV.** This shows that $\mathbf{(A\cap B)'}$ and $\mathbf{A' \cup B'}$ are represented by the same region, IV.

Thus, $\mathbf{(A\cap B)' = A' \cup B'}$

Practice Exercises

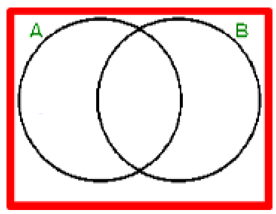
1. Complete the following Venn diagram using the information on the right.



All the whole numbers from 1 to 10 are to be included.

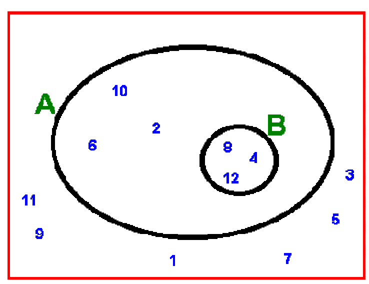
Set A = { 1 , 4 , 5 , 7 , 8 }
Set B = { 2 , 6 , 8 , 10 }

2. Complete the Venn diagram using the information on the right.



All the whole numbers from 1 to 10 are to be included.

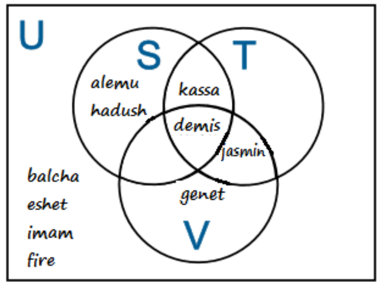
Set A contains all the odd numbers in this set.
Set B contains all the numbers greater than 4.



3. The whole numbers from 1 to 12 are included in the Venn diagram below.

   A. List set B
      (b) List set A
      (c) Which set contains all the even numbers?
   B. (d) A \ B
   C. (e) B

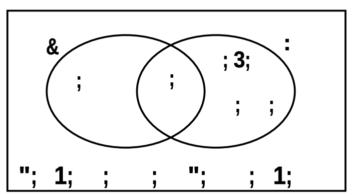
4. Consider below the Venn diagram for the ten friends example given earlier.



From the Venn diagram given list the elements of:

a) (S T) V?
b) S T
c) S V
d) T V
e) S (T V)
f) T (S U V)

5. Using the following Venn diagram:



a) List the elements that are
   I. in set A
   II. not in set B
   III. in $A \cap B$
   IV. $A' \cap B$
b) How many elements are there in:
   I. $A \cup B$
   II. $A \cap B'$

" $% & & '

!

- Sketch graphs of relations
- Identify the domain and range of a relation.
- Show relations as mappings and sets.

Suggested ways of teaching this topic: Guided Practice

Starter Activities

Before discussing how to sketch graphs of relations with inequalities, the teacher could remind students the different methods of representing a relation using an example of the following type:

Example: How could we show the relation {(2,1), (-1,3), (0,4)} in different ways?

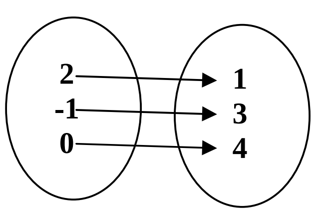
Solution:

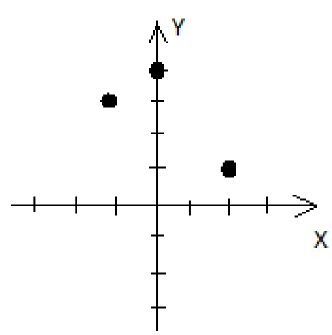
It can be shown by:

1) A table.

x	2	-1	0
y	1	3	4

2) A mapping.





3) A graph

Lesson Notes

The teacher should also have to make sure that students know how to sketch graphs of straight lines, which will be vital in sketching graphs of relations with inequalities, by asking students to sketch the graphs of some lines of the following type:

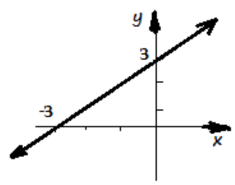
Example 1: Sketch the graph of the following functions:

A. $y = x + 3$
B. $y = 2 - x$
C. $y = 4$
D. $x = 2$

Solution: To sketch the graphs of these lines, it is possible to start with a table of values or finding x and y-intercepts –both of which students are familiar with at lower grades.

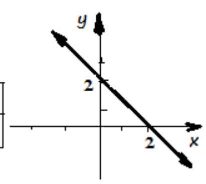
a) $y = x + 3$

x	-3	-2	-1	0	1	2
y	0	1	2	3	4	5



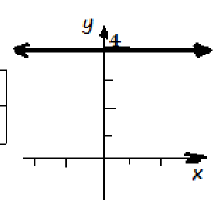
b) $y = 2 - x$

x	0	2
y	2	0



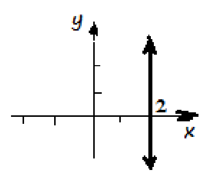
c) $y = 4$

x	-2	-1	0	1	2
y	4	4	4	4	4



d) $x = 2$

x	2	2	2	2	2
y	-2	-1	0	1	2



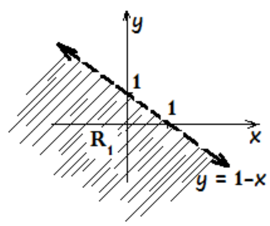
Now, it is the time to introduce graphs of relations with one inequality and step by step to two or more inequalities. Though students might not show it, grade 9 students are familiar with graphs of simple relations with one inequality at lower grades. Thus, it might be sufficient to give them some relations, of the following type, which could be done in pairs or small groups.

Example 2: Sketch the graphs of each of the following relations:

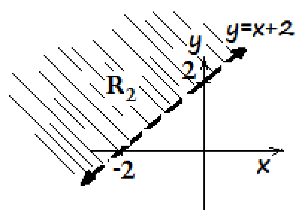
A. $R_1 = \{(x, y): y < 1 - x\}$
B. $R_2 = \{(x, y): y > x + 2\}$
C. $R_3 = \{(x, y): y \leq 3\}$
D. $R_4 = \{(x, y): y \geq - x - 2\}$

Solution:

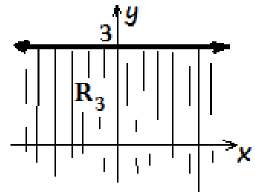
A. At the beginning, we need to find the boundary line. That is, y = $1 - x$. (Remember the inequality is strict less! So, the line must be broken) Then, we need to decide which way to shade. This can be checked by taking the origin, (0, 0) in to the inequality. For $y < 1 - x$, if we substitute (0, 0)thenwe get $0 < 1$,which is true. Thus, we shade below the line y = $1 - x$, towards the origin.



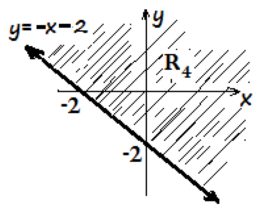
B. Here, the boundary line is, y = $x + 2$. As the inequality is strict greater, the line must be broken. Then, we need to decide which way to shade. This can be checked by taking the origin, (0, 0) in to the inequality. For $y > x + 2$, if we substitute (0, 0)thenwe get $0 > 2$,which is false. Thus, we shade above the line y = $x + 2$, opposite to the origin.



C. In R_3, the boundary line is, y = 3. As the inequality includes equality, the line must be solid. Then, we need to decide which way to shade. This can be checked by taking the origin, (0, 0) in to the inequality. For $y \leq 3$, if we substitute (0, 0)thenwe will get$0 \leq 3$,which is true. Thus, we shade below the line y = 3, towards to the origin.



D. In this case, the boundary line is, y = - *x* - *2*. As the inequality includes equality, the line must be solid. Then, we need to decide which way to shade. This can be checked by taking the origin, (0, 0) in to the inequality $y \geq - x - 2$. If we substitute (0, 0)then we will get $0 \geq -2$, which is true. Thus, we shade below the boundary line $y = - x - 2$, towards to the origin.

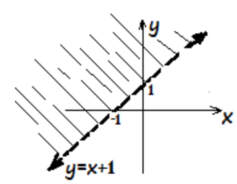


Example 3: Sketch the graphs of each of the following relations:

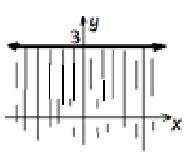
a) $R_1 = \{(x, y): y > x + 1 \text{ and } y \leq 3\}$
b) $R_2 = \{(x, y): y \geq x + 2 \text{ and } y < -x + 2\}$
c) $R_3 = \{(x, y): y > 1 - x, x \leq 2 \text{ and } y < 3\}$
d) $R_4 = \{(x, y): y > - x - 1, y < x + 1 \text{ and } x \leq 3\}$

Solutions:

a) Here, we have two boundary lines. y = *x*+*1 and y = 3*. Since the first inequality is strict less and the second includes equality, y = *x*+*1* must be broken and y = 3 must be solid. To decide which way to shade, we can check both inequalities, one by one, taking the origin, (0, 0).

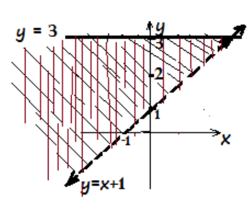


For $y > x + 1$, if we substitute (0, 0)then we get $0 > 1$, which is false. Thus, we shade above the line y = *x*+*1*, opposite to the origin, as shown.



For $y \leq 3$, if we substitute (0, 0)then we get $0 \leq 3$, which is true. Thus we shade below $y = 3$, towards the origin.

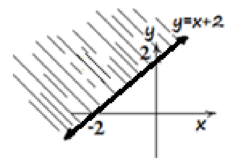
Since R_1 is a relation with two inequalities joined by "and", we will

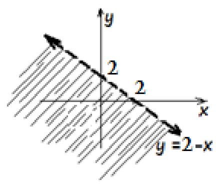


sketch both graphs on one co-ordinate plane and take the intersection of the two regions, as shown below.

b) Here, we have two boundary lines. $y = x + 2$ *and* $y = -x + 2$. Since the first inequality includes equality and the second inequality is strict less, $y = x + 2$must be a solid line and $y = -x + 2$must be broken. To decide which way to shade, we can check both inequalities, one by one, taking the origin, (0, 0).

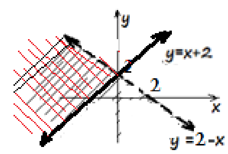
For $y \geq x + 2$, if we substitute (0, 0)thenwe get $0 \geq 2$, which is false. Thus, we shade above the line y = $x+2$, opposite to the origin, as shown.





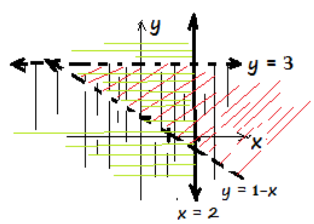
For $y < -x + 2$, if we substitute (0, 0)thenwe get $0 < 2$, which is true. Thus we shade below $y = -x+2$, towards the origin.

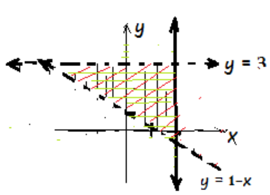
Since $R_2$is a relation with two inequalities joined by "and", we will take the intersection of the two shaded regions after graphing both on one co-ordinate plane.



c) In R_3, there are three boundary lines: y = $1 - x$, $x = 2$ and $y = 3$.
As the first and the last inequalities are strict, both lines y = $1 - x$ and $y = 3$ shall be solid. But, since the second includes equality the line $x = 2$must be solid.

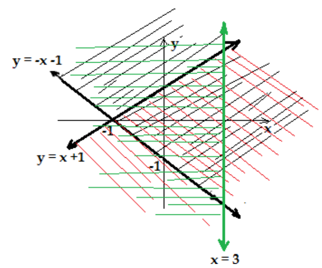
- For y $> 1 - x$, if we substitute (0, 0)thenwe get $0 > 1$, which is false. Thus, we shade above the line y =$1 - x$, opposite to the origin.
- For$x \leq 2$, whenwe substitute (0, 0)thenwe get $0 \leq 2$,which is true. Thus, we shade towards the origin, to the left of $x = 2$.
- *For* $y \leq 3$, if we substitute (0, 0)thenwe get $0 \leq 3$, which is true. Thus we shade below $y = 3$, towards the origin.
- Since R3 is a relation with three inequalities joined by "and", we will take the intersection of the three shaded regions after graphing both on one co-ordinate plane, as shown.
- The final regionR_3 will be the one shown below.



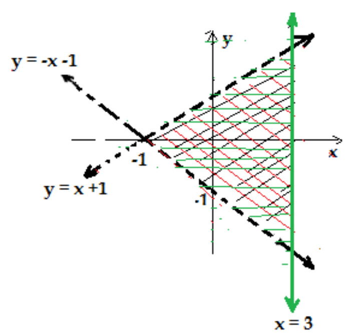


d) In this case, there are three boundary lines. They are: $y = -x - 1$, $y = x + 1$ and $x = 3$. As the first and second inequalities are strict, the lines $y = -x - 1$ and $y = x + 1$ must be broken. The third inequality includes equality, the line must be solid. Then, we need to decide which way to shade. This can be checked by taking the origin, (0, 0) in to the inequalities.

- For $y > -x - 1$, if we substitute (0, 0) then we will get $0 > -1$, which is true. Thus, we shade above the boundary line $y = -x - 1$, towards to the origin.
- For $y < x + 1$, if we substitute (0, 0) then we will get $0 < 1$, which is true. Thus, we shade below the boundary line $y = x + 1$, towards to the origin.
- For $x \leq 3$, if we substitute (0, 0) then we will get $0 \leq 3$, which is true. Thus, we shade towards the origin – to the left of the line $x = 3$.

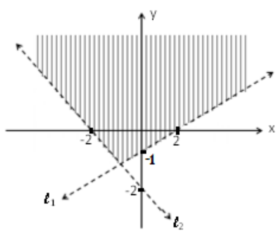


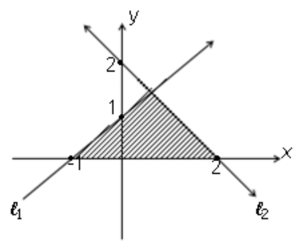
- The final region R_4 will be the one shown below.



Practice Exercises

1. Sketch the graphs of the following relations
   A. R = {(x, y): x + y < 2 and y < x}
   B. R = {(x, y): y > x -1, y > - x – 3 and y < 4}
   C. R = {(x, y): y   x + 1, x + y   -2 and x < 2}
2. Write the inequality represented by each of the following graphs





(#)

!

- Sketch graphs of relations
- Determine the domain and range of relations from their graphs

Suggested ways of teaching this topic: Guided Practice

Starter Activities

The teacher may start with asking questions of the following type:

“Who can define relations?”

“Who will tell us what domain means?”

“What about range?”

Expected Answers:

A relation is a set of ordered pairs

The domain is the set of 1st coordinates of the ordered pairs.

The range is the set of 2nd coordinates of the ordered pairs.

After the definitions are revised, the teacher may ask students to come up with answers to questions of the following type:

Given the relations: R_1= {(3,2), (1,6), (-2,0)} and R_2= {(2, 4), (3,-1), (0,-4), (3, 4)}find the domain and range of each relation.

Expected Answers:

For R_1 , Domain = {3, 1, -2} and Range = {2, 6, 0}

For R_2 , Domain = {2, 3, 0} and Range = {4, -1, -4}

Example 1: What is the domain of the relation {(2, 1), (4, 2), (3, 3), (4, 1)}

1. {2, 3, 4, 4}
2. {1, 2, 3, 1}
3. ✔ {2, 3, 4}
4. {1, 2, 3}
5. {1, 2, 3, 4}

Example 2: What is the range of the relation {(2, 1), (4, 2), (3, 3), (4, 1)}?

1. {2, 3, 4, 4}
2. {1, 2, 3, 1}
3. {2, 3, 4}
4. {1, 2, 3}
5. {1, 2, 3, 4}

After a short summary on the definitions of domain and range of relations, the teacher may continue the discussion, using appropriate examples of type given below, to answer the important question: “How could we determine the domain and range of relations with two or more inequalities?” (The graphs could be graphs of those relations discussed in Topic 4 above.

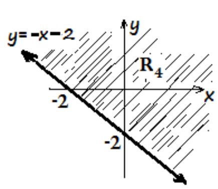
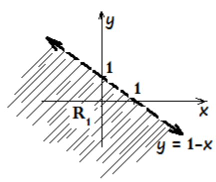
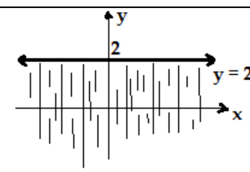
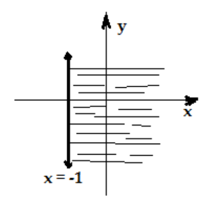
Major Idea:

To determine domain and range of relations with two or more inequalities, it is better to sketch their graph and see the region covered by the relation both horizontally and vertically.

The teacher may start with those graphs of relations with only one boundary line and continue with relations with two or more inequalities.

Lesson Notes

Example: Determine the domain and range of relations represented by the graphs given below

a) 	b) 	c) 	d) 

Solution:

a) Since the border line goes non-stop on both directions, there is no limit to the relation (shaded region) vertically and horizontally. Thus, the domain and range of this relation is the set of all real numbers
b) Same as in a)
c) As the boundary line does not limit the shaded region horizontally, the domain of the relation is the set of real numbers. But, as the boundary line limits the relation vertically at y = 2 and the line y = 2 is a solid one, the range of the relation is given by {y: y$\leq$ 2}.
d) As the boundary line limits the relation horizontally at x = -1 and the line is solid, the domain of the relation is given by {x: x $\geq$-1} But, as the boundary line does not limit the shaded region horizontally, the range of the relation is the set of real numbers.

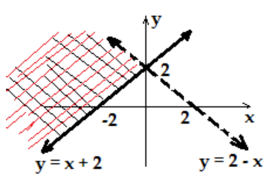
CONCLUSION

The teacher might ask students to conclude on the domain and range of the relations given by graphs of type a) and b) above. The students are expected to conclude as follows:

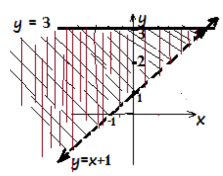
If the relation is represented by a graph with shaded region above or below one boundary line which is neither vertical nor horizontal straight line, the domain and range of the relation will be the set of all real numbers.

Example: Determine the domain and range of the relations represented by the graphs below.

a)



b)



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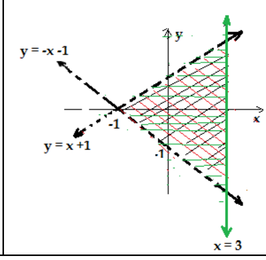

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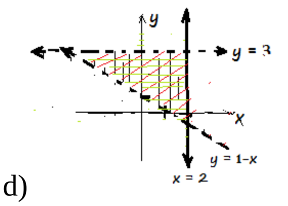


Solution:

a) Since the boundary lines y = 2 – x and y = x + 2 meet at (0, 2), with y= 2 – x a broken line, and the shaded region is limited horizontally to the right by the line x = 0, the domain of the relation is {x: x < 0}. But, as the region is not limited vertically, the range of the relation is the set of all real numbers.

b) The first step is to determine the intersection point of the boundary lines. To get that, we can solve both equations simultaneously using substitution method. Thus, substituting the value of y = 3 into y= x +1, we get 3 = x +1, which gives x = 2. So, the intersection point of the boundary lines is at (2, 3).

When we see the shaded region, it is limited horizontally to the right by the line x = 2, with the line y = x + 1 broken. So, the domain of the relation is {x: x < 2}. The shaded region is also limited vertically upwards by the solid line y = 3. So, the range of the relation is {y: y ≤ 3}.

c) Observing the shaded region, we see that it is triangular form, where it is bounded horizontally as well as vertically from both directions. Horizontally, the region is bounded by a broken line x = -1 from the left and by a solid line x = 3 from the right. Thus, the domain of the relation is {x: - 1 < x ≤ 3}. To get the boundary lines which limited the shaded region vertically, we need to find the intersection points of x = 3 with the line y = x + 1 and y = - x -1. This can be done by substitution. If we substitute x = 3 in y = x + 1and y = - x -1, we get y = 4 and y = - 4 respectively. Therefore, the range of the relation is {y: -4 < y < 4}.

Note:

Students must be reminded about the type of relationship between the strict less or greater symbols and broken the lines as well as why that relationship existed.

d) This shaded region is also in a triangular shape, bounded both horizontally and vertically. We need the intersection points of the lines x = 2 and y = 3 with the line y = 1– x. So, substitution results in y = -1 and x = – 2 respectively. So, the intersection points are (-2, 3) and (2, -1). Thus, the domain of the relation is {x: -2 < x < 2} and its rangeis {y: -1 < y < 3}.

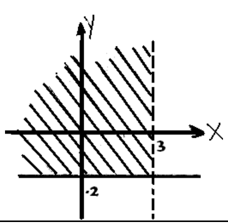
Concluding Activities

As discussed above, determining the domain and range of relations given their graphs is to see where the relation is limited vertically or horizontally or both. Students must be given relations without graphs, say R = {*(x, y): y ≤x - 2, y< -x+3 and y > -4*}, and shall be asked to determine the domain and range. (Of course, they need to sketch graphs!).Therefore, the teacher shall make sure that students have become capable of sketching graphs of relations and can write the domain and range from the graphs giving them relations of the above type and guiding them to practice and master the skills.

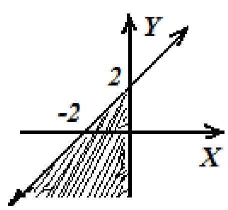
Practice Exercises

1. Write the inequality represented by each of the following graphs and determine their domain and range

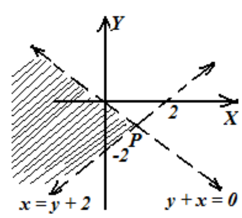
a)



b)



c)



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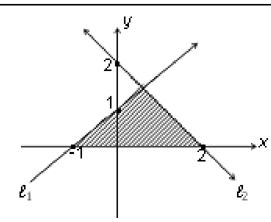

2. Sketch the graphs of the following relations and determine their domain and range.

A. R = {(x, y): y > x -1 and x > 2}
B. R = {(x, y): x + y 2 and y < x}
C. R = {(x, y): y x + 1, x + y -2 and x < 2}
D. R = {(x, y) = y <x + 2 and x + y < 0}
E. R = {x, y) = x<y+2 and y +x < 0}

* #

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- Determine the domain of functions
- Determine the range of functions

Methods of Teaching the Topic: Group discussion including guided practice

Starter Activities

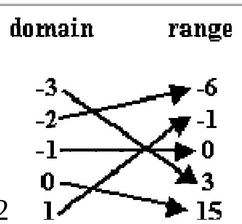
Focus and Review: The teacher shall remind students what has been learned in previous lessons that will be pertinent to this lesson and/or have them begin to think about the words and ideas of this lesson. Ask the students if they remember:

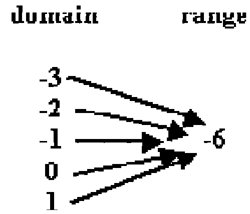
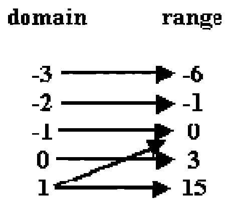
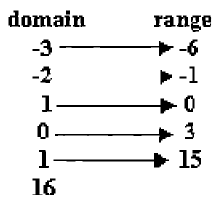
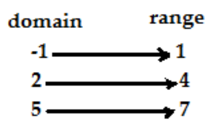
- What relations are and what functions are.
- The meaning of domain and range of relations
- The definition of functions
- To identify a given graph represents a function or not
- The vertical line test and what it is used for

Expected Explanations

A "relation" is just a relationship between sets of information. Think of all the people in one of your classes, and think of their heights. The pairing of names and heights is a relation. In relations and functions, the pairs of names and heights are "ordered", which means one comes first and the other comes second. To put it another way, we could set up this pairing so that you give me a name, and then I give you that person's height, or else you give me a height, and I give you the names of all the people who are that tall. The set of all the starting points is called "the domain" and the set of all the ending points is called "the range." The domain is what you start with; the range is what you end up with. The domain is the *x*'s; the range is the *y*'s.

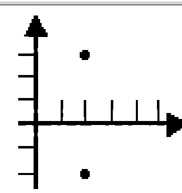
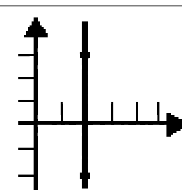
A function is a "well-behaved" relation. Just as with members of your own family, some members of the family of pairing relationships are better behaved than other. This means that, while all functions are relations. But, *not* all relations are functions. When we say that a function is "a well-behaved relation", we mean that, given a starting point, we know exactly where to go; given an *x*, we get only and exactly one *y*. The teacher may use the following examples to explain what function means.

<table>
<tr>
<td>2 
</td>
<td>This is a function. You can tell by tracing from each x to each y. There is only one y for each x; there is only one arrow coming from each x.</td>
</tr>
</table>

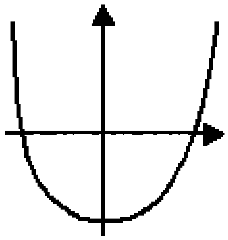
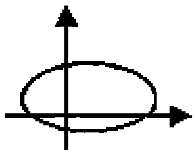
	This *is* a function! There is only one arrow coming from each *x*; there is only one *y* for each *x*. It just so happens that it's always the same *y* for each *x*, but it is only that one *y*. So this is a function; it's just an extremely *boring* function!
	This one is not a function: there are *two* arrows coming from the number 1; the number 1 is associated with two *different* range elements. So this is a relation, but it is not a function.
	Okay, this one's a trick question. Each element of the domain that has a pair in the range is nicely well-behaved. But what about that 16? It *is* in the domain, but it has no range element that corresponds to it! This won't work! So then this is not a function. It is not even a relation!
	This one is straight forward. For one domain element we have only one range element and vice-versa. It is a one to one function.

The "Vertical Line Test"

To continue with the discussion on functions graphically, the teacher might pause questions of the following type to be answered by a small group discussion: "what if we sketch the graph of the relation that consists of a set containing just two points: {(2, 3), (2, –2)} and test it with the vertical line test?" We already know that this is not a function, since $x = 2$ goes to each of $y = 3$ and $y = –2$.

If we graph this relation, it looks like:	
Notice that you can draw a vertical line through the two points, like this:	

This characteristic of non-functions was noticed and was codified in "The Vertical Line Test": Given the graph of a relation, if you can draw a vertical line that crosses the graph in more than one place, then the relation is not a function. Here are a couple examples:

	This graph shows a function, because there is no vertical line that will cross this graph twice.
	This graph does not show a function, because any number of vertical lines will intersect this oval twice. For instance, the *y*-axis intersects (crosses) the oval curve twice.

Lesson Notes

"Is it a function?" - Quick answer without the graph

Think of all the graphing that you've done so far. The simplest method is to solve for "y =", make a T-chart (table of values), pick some values for x, solve for the corresponding values of y, plot your points, and connect the dots. Not only is this useful for graphing, but this methodology gives yet another way of identifying functions: If you can solve for "y =", then it's a function. For example, $2y + 3x = 6$ is a function, because you can solve for y:

$2y + 3x = 6$
$2y = -3x + 6$

$y = \frac{-3}{2}x + 3$

On the other hand, $y^2 + 3x = 6$ is not a function, because you can't solve for a *uniquey*:

$y^2 + 3x = 6$

$y^2 = -3x + 6$

$y = \pm\sqrt{-3x + 6}$

This is solved for "y =", but it's not *unique*. Do we take the positive square root, or the negative? So, in this case, the relation is not a function. (We can also check this by using our first definition from above. Think of "$x = -1$". Then, we get $y^2 - 3 = 6$, so $y^2 = 9$, and then y can be either –3 or +3. That is, if we did an arrow chart, there would be two arrows coming from $x = -1$.)

Functions: Domain and Range

Let's return to the discussion of domains and ranges. When functions are first introduced, you will probably have some simplistic "functions" and relations to deal with, being just sets of points. These won't be terribly useful or interesting functions and relations, but your text wants you to get the idea of what the domain and range of a function are. For instance:

- State the domain and range of the following relation. Is the relation a function?
  {(2, –3), (4, 6), (3, –1), (6, 6), (2, 3)}

  The above list of points, being a relationship between certain *x*'s and certain *y*'s, is a relation. The domain is all the *x*-values, and the range is all the *y*-values. To give the domain and the range, we just list the values without duplication:

Domain: {2, 3, 4, 6} and Range: {–3, –1, 3, 6}

Students could be reminded that it is customary to list these values in numerical order, but it is *not* required. Sets are called "unordered lists", so they can list the numbers in any order they feel like. Just should not duplicate.

While the given set does represent a relation (because *x*'s and *y*'s are being related to each other), they give two points with the same *x*-value: (2, –3) and (2, 3). Since $x = 2$ gives two possible destinations, then **this relation is not a function.**

Note that all we had to do to check whether the relation is a function,is to look for duplicate *x*-values. If we find a duplicate *x*-value, then the different *y*-values mean that we do *not* have a function.

Example:

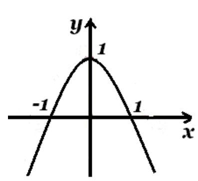
Given the relation:{(–3, 5), (–2, 5), (–1, 5), (0, 5), (1, 5), (2, 5)}, state the domain and range of the following relation. Is the relation a function?

Just list the *x*-values for the domain and the *y*-values for the range: So,

Domain: {–3, –2, –1, 0, 1, 2} and Range: {5}

In this example, every *x*-value goes to the exact same *y*-value. But each *x*-value is different, although it is boring; this relation is indeed a function. In point of fact, these points lie on the horizontal line $y = 5$.

Example: Is the following graph a function or not? If it is, write its domain and range.



Solution: It passes the vertical line test and so it is a function.

As the graph moves non- stop downward, its domain = **R** (all real numbers) and the Range is $\{y: y \leq 1\}$ = $(-\infty, 1]$

Example: Determine the domain and range of the following functions from their equations

a) $f(x) = |x - 1|$
b) $f(x) = x^2 - 4x + 3$
c) $f(x) = \begin{cases} x+1, & x<1 \\ 2, & x \geq 1 \end{cases}$

Solution:

a) $f(x) = |x - 1|$

All real numbers can fit into the value of x. Thus, Domain = **R**

But, for any value of x, the result y is always greater than or equal to 0. Thus, Range = $\{y: y \geq 0\} = [0, \infty)$

b) $f(x) = x^2 - 4x + 3$ using completing the square method, $f(x) = (x - 2)^2 - 1$

Domain = **R** Range = $[-1, \infty)$

c) $f(x) = \begin{cases} x+1, x<1 \\ 2, x \geq 1 \end{cases}$

The possible values of x is $x < 1$ in the first part and $x \geq 1$. Thus, the union is the set of all real numbers. So, Domain = **R**. For any real number x, the maximum we get is 2. Thus, Range = $(-\infty, 2]$

Domain of Composed functions: Sometimes some functions might be combinations of other functions. In that case the domain of such functions can be obtained by taking the intersection of the domains of the parts.

Example: Find the domain of $h(x) = \sqrt{3x} + x^2 - 1$.

Solution: $h(x) = \sqrt{3x} + x^2 - 1$ can be sum of two functions. $f(x) = \sqrt{3x}$ and $g(x) = x^2 - 1$

Domain of f = $[0, \infty)$ and Domain of g = **R**

Domain of h(x) = Domain of (f + g) (x) = Domain of f $\cap$ Domain of g

$= [0, \infty) \cap \mathbf{R} = [0, \infty)$

Example: Find the domain of $f(x) = \sqrt{x+4} + \frac{1}{x+4}$.

Solution:

f(x) = $\sqrt{x+4} + \frac{1}{x+4}$ can be expressed as a sum of two functions g(x) = $\sqrt{x+4}$ and r(x) = $\frac{1}{x+4}$

Domain of **g** = x ≥ - 4 or [-4, ∞)
Domain of r = **R**\ {-4}
Domain of f(x) = Domain of (g +r) (x) = [-4, ∞)∩**R**\ {-4}
= (-4, ∞)

Concluding Activities

The teacher shall make sure that the following are summarized by the students through exercises.

- A Relation R is a function if and only if whenever (x, y) ∈ R and (x, z) ∈ R then y = z.
- If a vertical line touches the graph of a relation at not more than one place, then the graph represents a function.
- The domain of the sum, difference, product and quotient of the two functions can be obtained from (Dom f) ∩ (Dom of g)

) +

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- sketch graphs of quadratic functions
- determine vertical axis and turning point of the graph of quadratic functions without sketching

Suggested ways of teaching this topic: Guided practice

Starter Activities

The teacher may begin with questions of the following type:

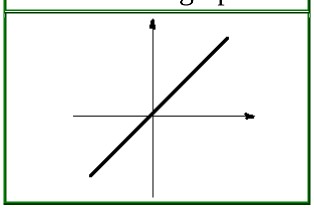
- What is the general form of quadratic equation?
- Who will show me completing the square for $f(x) = x^2 + 6x – 4$?
- Who can sketch the graph of $f(x) = 2x$?
- How do you find x –intercepts of a graph? For example: the x intercepts of $f(x) = 2x+1$
- Can anyone sketch the graph of $f(x) = x^2$

After thorough discussions of the ideas in the above questions, the teacher could lead students to the discussion of the graphs of quadratic equations starting from the basic quadratic is $y = x^2$.

Lesson Notes

The general technique for graphing quadratics is the same as for graphing linear equations. However, since quadratics graph as curvy lines (called "parabolas"), rather than the straight lines generated by linear equations, there are some additional considerations.

incorrect graph



The most basic quadratic function is $y = x^2$. When you graphed straight lines, you only needed two points to graph your line, though you generally plotted three or more points just to be on the safe side. However, three points will almost certainly *not* be enough points for graphing a quadratic function. For example, suppose a student computes these three points:

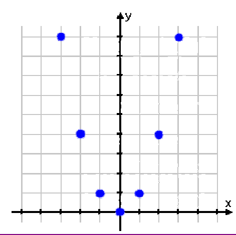
Then, based only on his experience with linear graphs, he tries to put a straight line through the points. He got the graph wrong.

x	$y = x^2$
0	$0^2 = 0$
1	$1^2 = 1$
2	$2^2 = 4$

Taking many points:

x	$y = x^2$
-3	$(-3)^2 = 9$
-2	$(-2)^2 = 4$
-1	$(-1)^2 = 1$
0	$0^2 = 0$
1	$1^2 = 1$
2	$2^2 = 4$
3	$3^2 = 9$
4	$4^2 = 16$

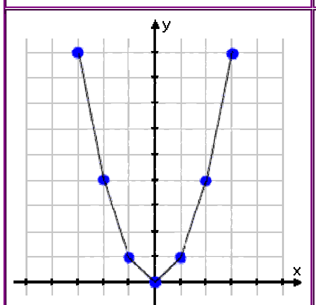
The last point has a rather large y-value (16), so we may decide that we won't bother drawing the graph large enough to plot it. Now, plot all the other points:



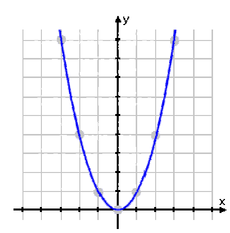
We draw a nicely smooth curving line neatly through the plotted points:

Some students may plot the points correctly, but will then connect the points with straight line segments, like the one in the box. Teacher should remind students the graph must be a smooth curve. They do still need a ruler for doing the graphing, but only for drawing the axes, not for drawing the parabolas. Parabolas graph as smoothly curved lines, not as jointed segments.

incorrect "segment" graph



correct graph of $y = x^2$



The general form $y = ax^2 + bx + c$

For graphing $f(x) = ax^2 + bx + c$, the leading coefficient "a" indicates how "fat" or how "skinny" the parabola will be.

For | a | > 1 (such as $a = 3$ or $a = -4$), the parabola will be "skinny", because it grows more quickly (three times as fast or four times as fast, respectively, in the case of our sample values of a).

For | a | < 1 (such as $a = {}^{1}/_{3}$ or $a = {}^{-1}/_{4}$), the parabola will be "fat", because it grows more slowly (one-third as fast or one-fourth as fast, respectively, in the examples). Also, if *a*is negative, then the parabola is upside-down.

We can see these trends when we look at how the curve $y = ax^2$ moves as "a" changes:As the leading coefficient goes from very negative to slightly negative to zero (not really a quadratic) to slightly positive to very positive, the parabola goes from skinny upside-down to fat upside-down to a straight line (called a "degenerate" parabola) to a fat right-side-up to a skinny right-side-up.

There is a simple, if slightly "dumb", way to remember the difference between right-side-up parabolas and upside-down parabolas. If, for instance, you have an equation where a is negative, but you're somehow coming up with plot points that make it look like the quadratic is right-side-up, then you will know that you need to go back and check your work, because *something* is wrong.

Parabolas always have a lowest point (if the parabola is up side) or a highest point (if the parabola is upside-down). This point, where the parabola changes direction, is called the "vertex".

If the quadratic is written in the form $y = a(x - h)^2 + k$, then the vertex is the point (h, k). This makes sense. The squared part is always positive (for a right-side-up parabola), unless it's zero. So we will always have that fixed value k, and then we will always be adding something to it to make y bigger, unless of course the squared part is zero. So the smallest y can possibly be is $y = k$, and this smallest value will happen when the squared part, $x - h$, equals zero. And the squared part is zero when $x - h = 0$, or when $x = h$.

The same reasoning works, with k being the largest value and the squared part always subtracting from it, for upside-down parabolas.

Note: The "a" in the vertex form "$y = a(x - h)^2 + k$" of the quadratic is the same as the "a" in the common form of the quadratic equation, "$y = ax^2 + bx + c$".

Since the vertex is a useful point, we can complete the square to convert $ax^2 + bx + c$ to vertex form, if not written in vertex form, for finding the vertex. But, it's simpler to just use a formula. (The vertex formula is derived from the completing the-square process, just as is the Quadratic Formula. In each case, memorization is probably simpler than completing the square.)

For a given quadratic $y = ax^2 + bx + c$, the vertex (h, k) is found by computing $h = \frac{-b}{2a}$, and then evaluating y at h to find k. Since students have already learned the Quadratic Formula, they may find it easy to memorize the formula for k. It is related to both the formula for h and the discriminate in the Quadratic Formula: $k = \frac{4ac - b^2}{4a}$.

Example: Find the vertex of $y = 3x^2 + x - 2$ and graph the parabola.

To find the vertex, we look at the coefficients a, b, and c. The formula for the vertex gives me:

$$h = {}^{-b}/_{2a} = {}^{-(1)}/_{2(3)} = {}^{-1}/_{6}$$

Then we can find k by evaluating y at $h = {}^{-1}/_{6}$:

$k = 3({}^{-1}/_{6})^2 + ({}^{-1}/_{6}) - 2$

$= {}^{3}/_{36} - {}^{1}/_{6} - 2$

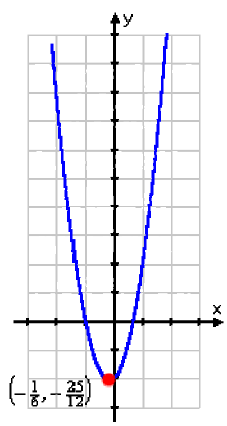
$= {}^{1}/_{12} - {}^{2}/_{12} - {}^{24}/_{12}$

$= {}^{-25}/_{12}$

So, we know that the vertex is at $({}^{-1}/_{6}, {}^{-25}/_{12})$. Using the formula was helpful, because this point is not one that we were likely to get on the T-chart. We need additional points for our graph:

x	$y = 3x^2 + x - 2$
-2	$3(-2)^2 + (-2) - 2 = 12 - 4 = 8$
-1	$3(-1)^2 + (-1) - 2 = 3 - 3 = 0$
0	$3(0)^2 + (0) - 2 = 0 - 2 = -2$
1	$3(1)^2 + (1) - 2 = 4 - 2 = 2$
2	$3(2)^2 + (2) - 2 = 14 - 2 = 12$

Now we can draw the graph, and we will label the vertex:



The only other consideration regarding the vertex is the "axis of symmetry". If we look at a parabola, we will notice that we could draw a vertical line right up through the middle which would split the parabola into two mirrored halves. This vertical line, right through the vertex, is called the axis of symmetry. If we are asked to determine the axis, we simply write down the line "$x = h$", where h is just the x-coordinate of the vertex. So in the example above, then the axis would be the vertical line $x = h = {}^{-1}/_{6}$.

Note: If the quadratic function has x-intercepts, a shortcut for finding the axis of symmetry is to note that this vertical line is always exactly between the two x-intercepts. So we can just take the average of the two intercepts to get the location of the axis of symmetry and the x-coordinate of the vertex.

Example: Find the vertex and intercepts of $y = 3x^2 + x - 2$ and graph; remember to label the vertex and the axis of symmetry.

Solution: We can find the vertex easily using the formula. But, we shall find the intercepts before we draw the graph. To find the y-intercept, we set x equal to zero and solve:

$$y = 3(0)^2 + (0) - 2 = 0 + 0 - 2 = -2$$

Then the y-intercept is the point (0, –2). To find the x-intercept, we set y equal to zero, and solve:

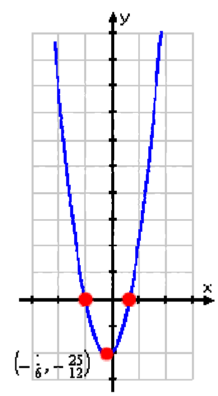
$0 = 3x^2 + x - 2$
$0 = (3x - 2)(x + 1)$
$3x - 2 = 0$ or $x + 1 = 0$
$x = {}^2/_3$ or $x = -1$

Then the x-intercepts are at the points (–1, 0) and $({}^2/_3, 0)$.

The axis of symmetry is halfway between the two x-intercepts at (–1, 0) and at $({}^2/_3, 0)$; using this, we can confirm the answer from the previous page:

$$(-1 + 2/3) / 2 = (-1/3) / 2 = -1/6$$

The complete answer is a listing of the vertex, the axis of symmetry, and all three intercepts, along with a nice neat graph:



The vertex is at $({}^{-1}/_6, {}^{-25}/_{12})$, the axis of symmetry is the line $x = {}^{-1}/_6$, and the intercepts are at (0, –2), (–1, 0), and $({}^2/_3, 0)$.

Example: Find the intercepts, the axis of symmetry, and vertex of $y = x^2 - x - 12$.

Solution: To find the y-intercept, we set x equal to 0 and solve:

$$y = (0)^2 - (0) - 12 = 0 - 0 - 12 = -12$$

To find the x-intercept, we set y equal to 0 and solve:

$0 = x^2 - x - 12$
$0 = (x - 4)(x + 3)$
$x = 4$ or $x = -3$

To find the vertex, we look at the coefficients: $a = 1$ and $b = -1$. Inserting into the formula, we get:

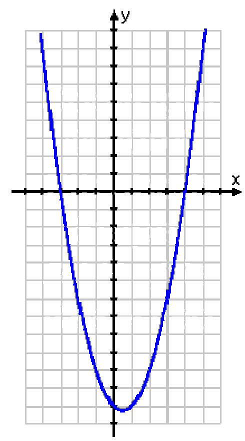
$$h = {}^{-(-1)}/_{2(1)} = {}^{1}/_{2} = 0.5$$

To find k, we replace $h = {}^{1}/_{2}$ for x in $y = x^2 - x - 12$, and simplify:

$$k = ({}^{1}/_{2})^2 - ({}^{1}/_{2}) - 12 = {}^{1}/_{4} - {}^{1}/_{2} - 12 = -12.25$$

Once we have the vertex, it's easy to write down the axis of symmetry: $x = 0.5$. Now, we will find some additional plot points, to fill in the graph:

x	$y = x^2 - x - 12$
-4	$(-4)^2 - (-4) - 12 = 16 + 4 - 12 = 8$
-3	$(-3)^2 - (-3) - 12 = 9 + 3 - 12 = 0$
-2	$(-2)^2 - (-2) - 12 = 4 + 2 - 12 = -6$
-1	$(-1)^2 - (-1) - 12 = 1 + 1 = 12 = -10$
0	$(0)^2 - (0) - 12 = 0 - 0 - 12 = -12$
1	$(1)^2 - (1) - 12 = 1 - 1 - 12 = -12$
2	$(2)^2 - (2) - 12 = 4 - 2 - 12 = -10$
3	$(3)^2 - (3) - 12 = 9 - 3 - 12 = -6$
4	$(4)^2 - (4) - 12 = 16 - 4 - 12 = 0$
5	$(5)^2 - (5) - 12 = 25 - 5 - 12 = 8$



For convenience, we may pick x-values that were centered on the x-coordinate of the vertex. Now we can plot the parabola as shown above and the final answer to our questions are:

- The vertex is at the point (0.5, –12.25),
- The axis of symmetry is the line x = 0.5,
- The intercepts are at the points (0, –12), (–3, 0), and (4, 0).

Example: Find the x-intercepts and vertex of $y = -x^2 - 4x + 2$.

Solution:

Since it is so simple to find the y-intercept, they are only asking for the x-intercepts this time. To find the x-intercept, we set y equal 0 and solve:

$0 = -x^2 - 4x + 2$
$x^2 + 4x - 2 = 0$

$$
\begin{aligned}
x &= \frac{-(4) \pm \sqrt{(4)^2 - 4(1)(-2)}}{2(1)} \\
&= \frac{-4 \pm \sqrt{16+8}}{2} \\
&= \frac{-4 \pm \sqrt{24}}{2} \\
&= \frac{-4 \pm 2\sqrt{6}}{2} \\
&= -2 \pm \sqrt{6}
\end{aligned}
$$

To find the vertex, we look at the coefficients: $a = -1$ and $b = -4$. Then:

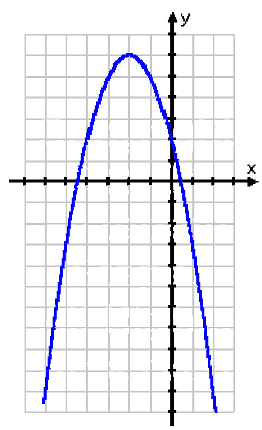
$$h = \frac{-(-4)}{2(-1)} = -2$$

To find k, we replace $h = -2$ in for x in $y = -x^2 - 4x + 2$, and simplify:

$k = -(-2)^2 - 4(-2) + 2 = -4 + 8 + 2 = 10 - 4 = 6$

Now we will find some additional plot points, to help us sketch the graph:

x	$y = -x^2 - 4x + 2$
-6	$-(-6)^2 - 4(-6) + 2 = -36 + 24 + 2 = -10$
-5	$-(-5)^2 - 4(-5) + 2 = -25 + 20 + 2 = -3$
-4	$-(-4)^2 - 4(-4) + 2 = -16 + 16 + 2 = 2$
-3	$-(-3)^2 - 4(-3) + 2 = -9 + 12 + 2 = 5$
-2	$-(-2)^2 - 4(-2) + 2 = -4 + 8 + 2 = 6$
-1	$-(-1)^2 - 4(-1) + 2 = -1 + 4 + 2 = 5$
0	$-(0)^2 - 4(0) + 2 - 0 - 0 + 2 - 2$
1	$-(1)^2 - 4(1) + 2 = -1 - 4 + 2 = -3$
2	$-(2)^2 - 4(2) + 2 = -4 - 8 + 2 = -10$



Note that we picked x-values that were centered on the x-coordinate of the vertex. Thus,

- The vertex is at $(-2, 6)$,
- The x intercepts are points: $(-2-\sqrt{6})$ and $(-2+\sqrt{6})$

Example: Find the x-intercepts and vertex of $y = -x^2 + 2x - 4$.

To find the vertex, I look at the coefficients: $a = -1$ and $b = 2$. Then:

$$h = \frac{-(2)}{2(-1)} = 1$$

To find k, replace $h = 1$ for x and simplify:

$$k = -(1)^2 + 2(1) - 4 = -1 + 2 - 4 = 2 - 5 = -3$$

The vertex is below the *x*-axis, and, since the leading coefficient is negative, the parabola is going to be upside-down. So can the graph possibly cross the *x*-axis? Can there possibly be any *x*-intercepts? Of course not! So we expect to get "no (real) solution" when we try to find the *x*-intercepts, but we shall show the work anyway. To find the *x*-intercept, we set *y* equal 0 and solve:

$$0 = -x^2 + 2x - 4$$

$$x^2 - 2x + 4 = 0$$

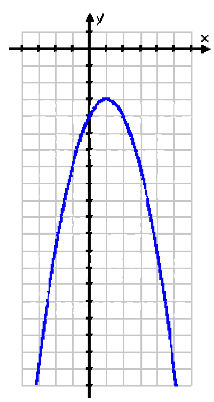
$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(4)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm \sqrt{-12}}{2}$$

$$= \frac{2 \pm 2\sqrt{-3}}{2} = 1 \pm \sqrt{-3}$$

As soon as we get a negative inside the square root, we know that we can't get a solution. So, as expected, there are no *x*-intercepts. Now, let's find some additional plot points, to draw the graph:

x	$-x^2+2x-4$
-3	$-(-3)^2+2(-3)-4=-9-6-4=-19$
-2	$-(-2)^2+2(-2)-4=-4-4-4=-12$
-1	$-(-1)^2+2(-1)-4=-1-2-4=-7$
0	$-(0)^2+2(0)-4=0+0-4=-4$
1	$-(1)^2+2(1)-4=-1+2-4=-3$
2	$-(2)^2+2(2)-4=-4+4-4=-4$
3	$-(3)^2+2(3)-4=-9+6-4=-7$
4	$-(4)^2+2(4)-4=-16+8-4=-12$
5	$-(5)^2+2(5)-4=-25+10-4=-19$



Therefore, the vertex is at (1, –3), and the only intercept is y intercept at (0, –4).

Concluding Activities

The teacher shall make sure that the following summary points of the topic are well taken by the students. The teacher could use a short quiz or a test to evaluate students' progress apart from observation while guiding the practice.

- The graph of a quadratic function, $f(x) = ax^2 + bx + c$, $a \neq 0$, is a parabola which turns up if $a > 0$ and down if $a < 0$.

- The turning point of the graph of a quadratic function $f(x) = ax^2 + bx + c, a \neq 0$ is called the vertex. It can be obtained from $\frac{-b}{2a}, \frac{4ac-b^2}{4a}$
- The vertical line through the vertex (turning point) is called the axis of symmetry. It is the vertical line $x = \frac{-b}{2a}$
- Any quadratic function $f(x) = ax^2+bx+c, a \neq 0$ can be written as $f(x) = a(x+d)^2 + e$ using completing the square method.

Practice Exercises

1. Rewrite $f(x) = x^2 + 2x - 24$ in the form $f(x) = (x+d)^2 + e$, where d, e, $\in$ **R.** Indicate the vertex & sketch the graph.
2. Rewrite $f(x) = 2x^2 - 3x - 6$ in the form $f(x) = a(x+d)^2 + e$ and find the vertex.
3. What do we do to get the graphs of the following from the graph of $f(x) = x^2$.

   i. $g(x) = x^2 + 5$
   ii. $g(x) = x^2 - 5$
   iii. $g(x) = (x - 5)^2$
   iv. $g(x) = (x + 5)^2$

4. Determine the intercepts, vertex and axis of symmetry for each of the following quadratic functions and then sketch their graphs.

   i. $f(x) = -3x^2$
   ii. $f(x) = -x^2 - 5$
   iii. $f(x) = -x^2 + 5$

" $,

!

- State similarity triangle theorems
- Apply similarity triangle theorems
- Apply similarity ratio on similar triangles

Suggested ways of teaching this topic: Guided practice, Q&A, small group discussion,…

$

The teacher may start with the brainstorming question like:

- Who can tell me the definition of similar triangles? Or
- When do we say two triangles are similar?
- What kind of symbol do we use to denote similarity?
- Who can demonstrate the similarity of two triangles in drawing?

Expected Answer

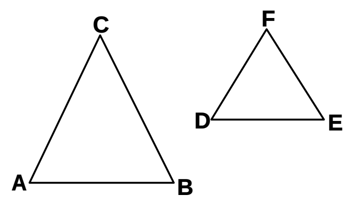
Triangles are similar if their corresponding (matching) angles are congruent (equal) and the corresponding sides are in same proportion.

The Symbol for Similar is: $\sim$

When we say $\Delta ABC \sim \Delta DEF$,

The 3 corresponding angles are congruent

i.e. $\angle A \equiv \angle D$, $\angle B \equiv \angle E$, $\angle C \equiv \angle F$



ΔABC ~ ΔDEF

The 3 corresponding sides are in proportion

i.e. $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$

The teacher shall ask the following question to make sure students understood the idea of proportional sides in similar triangles.

- If $\Delta ABC \sim \Delta XYZ$ then list down as many proportional equations as you can.

Expected Answer:

ΔABC ~ ΔXYZ $\dfrac{AC}{XZ} = \dfrac{AB}{XY} = \dfrac{BC}{YZ}$

ΔABC ~ ΔXYZ $\dfrac{AC}{AB} = \dfrac{XZ}{XY}$

ΔABC ~ ΔXYZ $\dfrac{AC}{BC} = \dfrac{XZ}{YZ}$

ΔABC ~ ΔXYZ $\dfrac{AB}{BC} = \dfrac{XY}{YZ}$

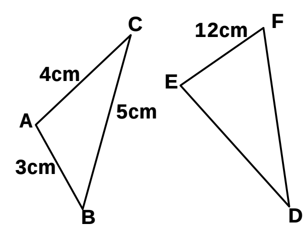
What is similarity ratio?

The similarity ratio is the proportion that you get when you divide any corresponding side of one triangle by its corresponding side in other similar triangle.

Lesson Notes

The teacher shall use the following examples to let students have a better practice on finding ratio of sides in similar triangles.

Example: Let ΔABC ~ ΔEFD as shown in the figure. Find the length of $\overline{ED}$ *and* $\overline{FD}$

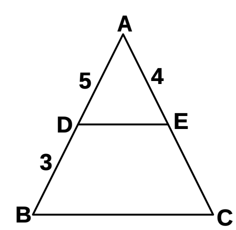


Solution: ΔABC ~ ΔEFD $\dfrac{AC}{AB} = \dfrac{ED}{EF}$

$\dfrac{ED}{12} = \dfrac{4}{3}$ ED = $\dfrac{12 \times 4}{3}$ = 16cm

ΔABC ~ ΔEFD $\dfrac{AB}{BC} = \dfrac{EF}{FD}$ $\dfrac{FD}{BC} = \dfrac{EF}{AB}$

$\dfrac{ED}{5} = \dfrac{12}{3}$ FD = $\dfrac{5 \times 12}{3}$ = 20cm



Example: In the figure, if ΔABC ~ ΔADE, What is the length of AC?

Solution: $\Delta ABC \sim \Delta ADE$ $\frac{AC}{AE} = \frac{AB}{AD}$

$\frac{EC+4}{4} = \frac{8}{5}$ $EC + 4 = \frac{4\times 8}{5} = \frac{32}{5}$

EC = 6.4 – 4 = 2.4cm

AC = AE + EC

= 4 + 2.4 = 6.4cm

The teacher should ask students about the relationship between congruent and similar triangles. The following example might help.

Example: Prove that if $\Delta ABC \equiv \Delta DEF$, then $\Delta ABC \sim \Delta DEF$

Proof: $\Delta ABC \equiv \Delta DEF$ AB = DE, BC = EF and AC = DF

$\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF} = 1$

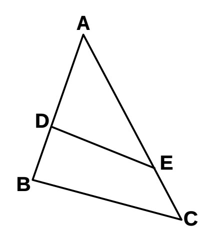
That is, if two triangles are congruent then they are similar.

Teacher can ask students: “Is the converse statement true?”

Expected Answer

In similar triangles, only the angles are congruent but the sides are not necessarily congruent. Therefore, similar triangles may not be congruent.

Property of a line parallel to a side of a triangle



If a line parallel to a side of a triangle is drawn between the side and the opposite angle, then it divides the other two sides proportionally

That is, $\frac{AD}{AB} = \frac{AE}{AC}$ or $\frac{AD}{BD} = \frac{AE}{EC}$

After a thorough discussion on the similarity ratio, the teacher may guide students with the following similarity theorems, their proofs and applications.

The AAA Theorem

If all the three angles of ΔABC is congruent to the corresponding angles of ΔDEF,

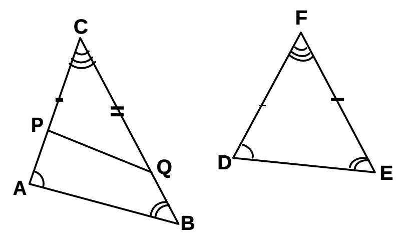
$\Delta ABC \sim \Delta DEF$

Proof

Given: ΔABC and ΔDEF with $\angle A \equiv \angle D$, $\angle B \equiv \angle E$ and $\angle C \equiv \angle F$

To prove: ΔABC ~ ΔDEF

Assume that AC > DF and BC > EF and mark points P and Q on $\overline{AC}$ *and* $\overline{BC}$ such that PC = DF and QC = EF

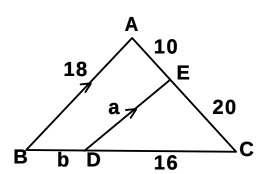


Statement	*Reason*
1. $\overline{PC} \equiv \overline{DF}$	1. Construction
2. $\angle C \equiv \angle F$	2. Given
3. $\overline{QC} \equiv \overline{EF}$	3. Construction
4. ΔPCQ ≡ΔDFE	4. SAS postulate
5. $\angle CPQ \equiv \angle D$ and $\angle CQP \equiv E$	5. Corresponding angles of congruent triangles
6. $\angle CPQ \equiv \angle A$ and $\angle CQP \equiv \angle B$	6. Transitivity of congruence of angles
7. $\overline{PQ} \equiv \overline{DE}$	7. Corresponding sides of congruent triangles
8. $\frac{PC}{AC} = \frac{QC}{BC}$	8. Property of a line parallel to side of a triangle
9. $\frac{DF}{AC} = \frac{FE}{BC}$	9. Substitution

Similarly, $\frac{DE}{AB} = \frac{EF}{BC}$

Since the corresponding angles are congruent and the corresponding sides are proportional, we have ΔABC ~ ΔDEF.

Example: In the figure, if $\overline{AB} \parallel \overline{ED}$ Calculate a, b and c



Solution: ECD ACB by AAA

$$\frac{AB}{CA} = \frac{ED}{CE}$$

$$\frac{18}{30} = \frac{a}{20} \qquad a = \frac{18 \times 20}{30} \qquad a = 12 \qquad \frac{BD}{DC} = \frac{AE}{EC} \qquad \frac{b}{16} = \frac{10}{20} \qquad b = \frac{16 \times 10}{20} = 8$$

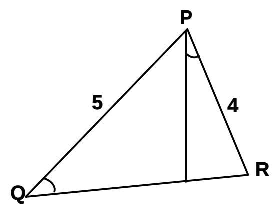
Theorem: (AA theorem)

If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.

The teacher could leave the proof to the students giving the clue that they can use the previous AAA Theorem.(Of course, they shall use angle sum theorem of triangles!)

Example: In the figure, if PQ = 5, QR = 6, PR = 4 and $\angle PQR \equiv \angle RPK$, find KR and KP.

Solution:



Statement	*Reason*
1. $\angle R \equiv \angle R$	1. Common angle
2. $\angle PQR \equiv \angle RPK$	2. Given
3. $\Delta RQP \sim \Delta RPK$	3. AA similarity theorem

$$\frac{RP}{RQ} = \frac{RK}{RP} \qquad \frac{4}{6} = \frac{RK}{4} \qquad RK = \frac{16}{6} = \frac{8}{3}$$

$$\frac{PQ}{RQ} = \frac{KP}{RP} \qquad \frac{5}{6} = \frac{KP}{4} \qquad KP = \frac{5 \times 4}{6} = \frac{10}{3}$$

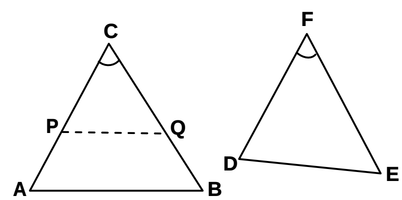
The SAS Similarity Theorem

If two sides of a triangle are proportional to two sides of another triangle and the angle included in these sides are congruent, then the triangles are similar.

Proof:

Given: ΔABC and ΔDEF with

$$\angle C \equiv F, \quad \frac{AC}{DF} = \frac{BC}{EF}$$



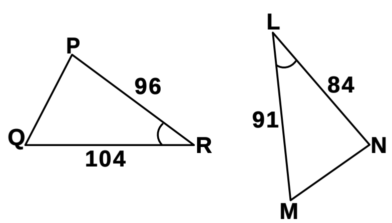
To prove: $\Delta ABC \sim \Delta DEF$

Construction: Assume AC > DF and BC > EF. Mark points P and Q on $\overline{AC}$ and $\overline{BC}$ respectively such that $\overline{AC} \equiv \overline{DF}$ and $\overline{QC} \equiv \overline{EF}$

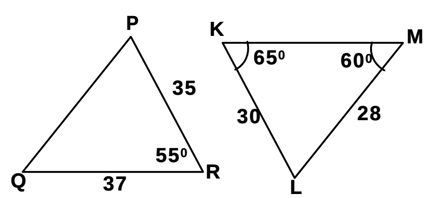
Statement	*Reason*
1. $\overline{PC} \equiv \overline{DF}$	1. Construction
2. $\angle C \equiv \angle F$	2. Given
3. $\overline{QC} \equiv \overline{EF}$	3. Construction
4. $\Delta PQC \equiv \Delta DEF$	4. SAS postulate
5. $\frac{AC}{DF} = \frac{BC}{EF}$	5. Given
6. $\frac{AC}{PC} = \frac{BC}{QC}$	6. Substitution
7. $\overline{PQ} \,//\, \overline{AB}$	7. property of propotional line segments
8. $\angle CPQ \equiv \angle A$, $\angle CQP \equiv \angle B$	8. Corresponding angles
9. $\angle D \equiv \angle A$, $\angle E \equiv \angle B$	9. Substitution
10. $\Delta ABC \sim \Delta DEF$	10. AA similarity theorem

Example: Are the given pairs of triangles similar?

a)



b)



Solution:

A) congruent angles: $\angle R \equiv \angle L$

Sides in proportion: $\frac{LN}{PR} = \frac{LM}{QR}$ $\quad \frac{84}{96} = \frac{91}{104} \Leftrightarrow \frac{7}{8} = \frac{7}{8}$

Therefore, $\Delta PQR \sim \Delta NML$ since $\angle R \equiv \angle L$ and $\frac{LN}{PR} = \frac{LM}{QR}$

B) $m(\angle L) = 180^0 - (65^0 + 60^0) = 55^0 = m(\angle R)$

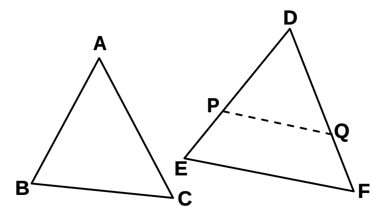
$\frac{PR}{ML} = \frac{35}{28}$ *and* $\frac{QR}{KL} = \frac{37}{30}$ $\quad \frac{PR}{ML} \neq \frac{QR}{KL}$ Therefore, ΔPQR is **not** similar to ΔKLM

The SSS Similarity Theorem

If three sides of one triangle are proportion to three sides of another triangle, then the two triangles are similar.

Proof

Given: ΔABC and ΔDEF, With $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$



To prove: ΔABC ~ ΔDEF

Construction: Assume that DE > AB and DF > AC.

Make point P on $\overline{DE}$ such that $\overline{DP} \equiv \overline{AB}$. Then draw a line parallel to $\overline{EF}$ meet DF at Q.

Statement	***Reason***
1. ∠DPQ ≡∠DEF	1. Corresponding angles
2. ∠D ≡∠D	2. Common angle
3. ΔDPQ ~ ΔDEF	3. AA similarity theorem.
4. $\frac{DP}{DE} = \frac{PQ}{EF} = \frac{DQ}{DF}$	4. Def. of similar triangles.
5. $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$	5. Given
6. $\overline{DP} \equiv \overline{AB}, \overline{PQ} = \overline{BC}, \overline{DQ} \equiv \overline{AC}$	6. Steps 4 & 5 and substitution
7. ΔABC ≡ΔDPQ	7. SSS theorem
8. ΔABC ~ ΔDEF	8. Step 3 and substitution

Concluding Activities

The teacher shall make sure that the following are summarized by the students. The applications of the theorems are also well practiced through teacher supervised exercises.

Two triangles are said to be similar if all their corresponding angles are congruent and the corresponding sides are proportional.

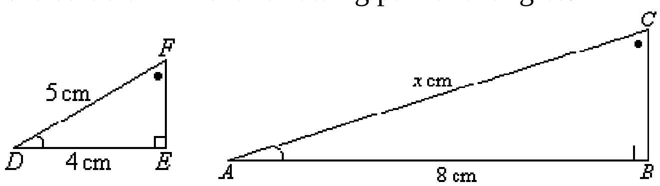
The AA similarity theorem: If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.

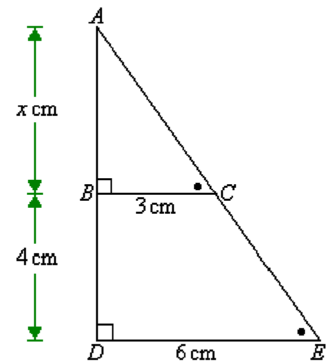
The SAS Similarity Theorem: If two sides of a triangle are proportional to two sides of another triangle and the angle included in these sides is congruent, then the triangles are similar.

The SSS Similarity Theorem: If three sides of one triangle are proportional to three sides of another triangle, then the two triangles are similar.
Congruent triangles are similar but similar triangles may not be congruent.

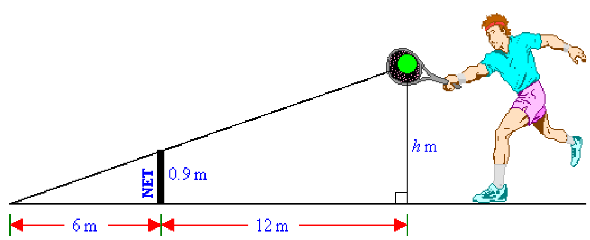
Practice Exercises

1. Find the value of *x* in the following pair of triangles.

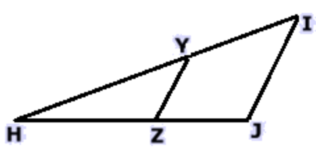




2. Find the value of the height, *h* m, in the following diagram at which the tennis ball must be hit so that it will just pass over the net and land 6 m away from the base of the net.



3. Below are two versions of HIZ and HYZ. Which of these two versions give a pair of similar triangles?



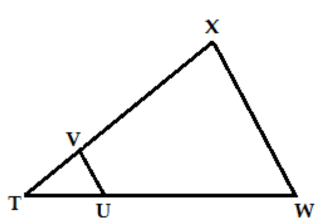
VERSION 1

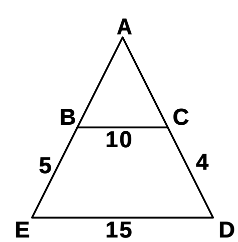
HY = 4	HI = 8
HZ = 6	IJ = 10
YZ = 6	ZJ = 6

VERSION 2

HY = 4	HI = 8
YZ = 5	IJ = 10
HZ = 6	ZJ = 2

4. In the figure, TUV and TWX are similar withTU = 10 and TW = 40, what is the similarity ratio?





5. In the following figure, ED // BC. A, B and E are collinear and A, C and D are also collinear. Find AC and AB

- !

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- state angle properties of circles
- apply angle properties of circles to solve related problems

Suggested ways of teaching this topic: Guided practice,

Starter Activities

The teacher could start with brainstorming questions like:

- Define the terms: chord, diameter, radius, tangent, secant, arc
- What is the relationship between diameter and any chord perpendicular to that diameter?

Do you agree with the following statements?

1) In Equal circles or in the same circle equal chords are equidistant from the center
2) Chords which are equidistant from the center are equal
3) Tangents from an external point are equal in length

Expected Answers

Chord: a line segment joining two points of a circle or a line segment whose end points lie on the circle.

Diameter: the largest chord through the center of a circle.

Radius: a line segment drawn from the center to any point on the circle.

Tangent: any line which touches the outer part of the circle at only one point.

Secant: any line that cuts a circle in to two.

The diameter bisects any chord perpendicular to it.

Yes! All the statements are correct.

Angle Properties of Circles

The teacher shall revise also on the meanings of major and minor arcs, angle subtended by an arc and arcs intercepted by an angle using short Q & A with students. Then, it would be appropriate time to discuss the following four angle properties of circles using guided practice:

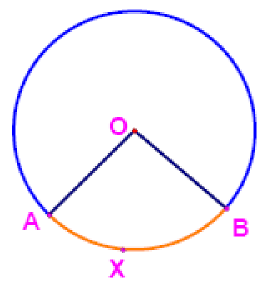
(1) Angle at Centre
(2) Angle in Semicircle
(3) Angles in Same Segment
(4) Angles in Opposite Segments

Lesson Notes

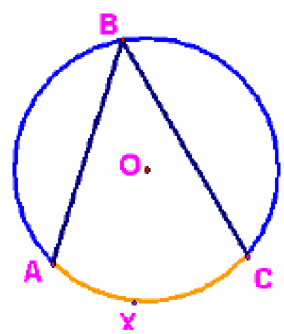
An **inscribed angle** is an angle whose vertex is on the circumference of a circle, and whose sides contain chords of the circle.

An angle intercepts an arc if:

- The END-POINTS of the arc lie on the angle
- Each SIDE of the angle contains at least one end -point of the arc
- Except for its end -points, the arc lies in the interior of the angle.

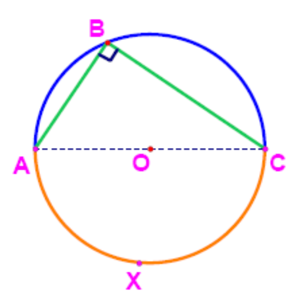


Theorem: The measure of an angle inscribed in a circle is half of the measure of the arc subtending it.



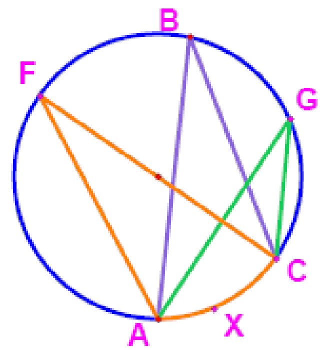
m(∢ABC)= ½ m(arc AXC)

Theorem: An angle inscribed in a semi -circle is a right angle. The converse of this corollary is that a circular arc in which a right angle is inscribed must be a semi -circle.



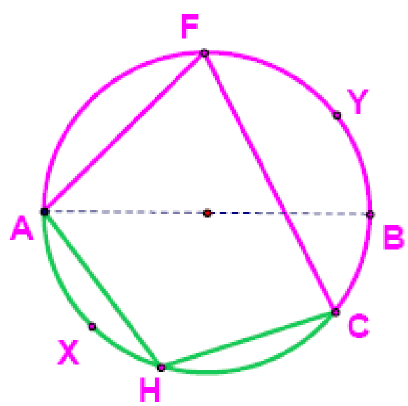
m(∢ABC)= ½ m(arc AXC) = 90^0

Theorem: Angles inscribed in the same arc must be congruent.

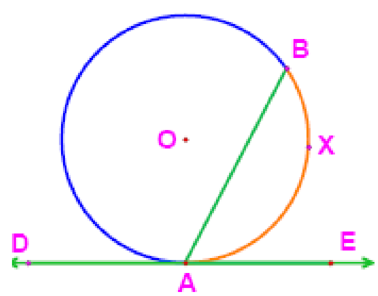


m($\angle$ABC)= m($\angle$AFB) =m($\angle$AGC)= = ½ m(arc AXC)

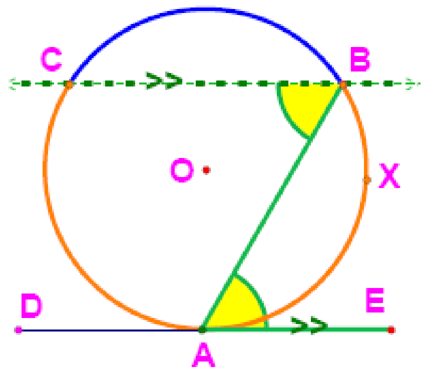
Theorem: An angle inscribed in an arc LESS than a semi -circle will be OBTUSE, i.e. greater than 90^0. An angle inscribed in an arc GREATER than a semi -circle will be ACUTE.



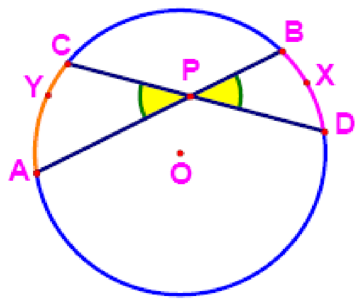
Theorem: The angle formed by a tangent and a chord drawn from the point of tangency is measured by half the arc it intercepts.



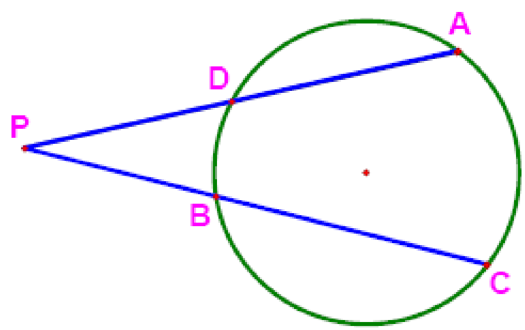
Theorem: The measure of the angle formed by two intersecting chords is half the sum of the measures of the arc intercepted by the angle and its vertically opposite angle.



Theorem: The point of intersection separates the chord into 2 segments. The product of the lengths of the segments for one chord is the same as the product for the other chord.

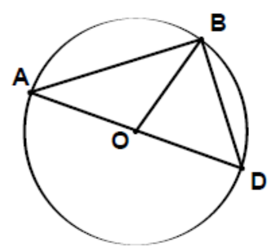


Theorem: The measure of an angle formed by 2 secants is one -half of the difference of the measure of the intercepted arcs.



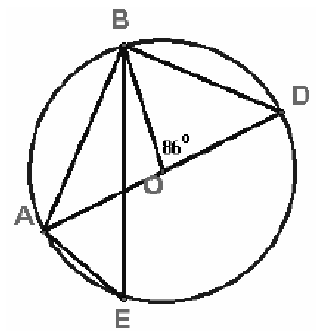
Practice Exercise

1. Given Circle O, as shown,



a) Name one minor arc in the circle.
b) Name one major arc in the circle.
c) Name the angle subtended at the centre by arc BD.
d) Name the inscribed angle subtended by arc BD.
e) An angle in a semi-circle.
f) 2 angles subtended by chord AB.

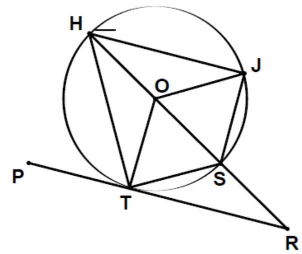
2. Given: Circle with centre O, as shown below, with m(DOB) = 86 °



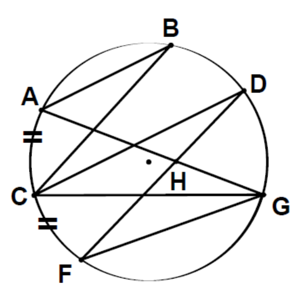
Calculate:

a) m(DAB)
b) m(AOB)
c) m(ABD)
d) m(ODB)
e) m(AEB)

3. Given circle with center O, as shown in the figure, if m(arc HT)=m(arc HJ) and m(PTH)= 58, calculate the measure of the remaining angles or arcs.



4. In the figure below m(arc AC)=m(arc CF), m(AGF)= 42^0 and m(GCF)= 25^0. Calculate



a) m(ABC)

b) m(DHG)

!

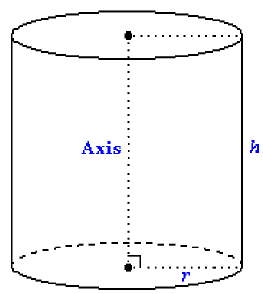
- Determine the surface area of different types of cylinders
- Calculate the volume of cylinders

Suggested ways of teaching this topic: teacher's explanation using demo, Q & A, guided practice, independent practice

Starter Activities

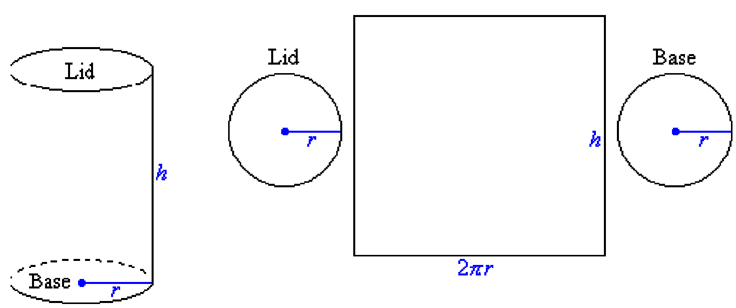
The teacher shall start explaining the terms: radius, axis, base, curved surface and total surface using demo on a cylindrical object as shown below.

Expected Explanation



Cylinder

The radius of the circular cross-section is called the **radius** of the cylinder, and the straight line that passes through the center of each circular cross-section is called the **axis** of the cylinder. The length of the axis is called the **height** of the cylinder. Consider a cylinder of radius *r* and height *h*, the radius and height of the cylinder are represented by *r* and *h* respectively. The total surface area (*TSA*) includes the area of the circular top and base, as well as the curved surface area (*CSA*).



To determine a formula for the **curved surface area** of a cylindrical can, wrap a sheet of paper snugly around the can and tape it together. Trim the paper at the top and bottom to match the shape of the can. Then slide the paper off the can and cut this paper cylinder parallel to its axis so that it forms the rectangle shown in the above diagram.

Lesson Notes

As canbe seen from the above drawings, the length of the rectangle = circumference of the base circle

$$= 2\ \ r$$

The width of the rectangle = height of the cylinder

$$= h$$

$$\therefore TSA = \text{Area of the Lid} + \text{Area of the Base} + \text{Curved Surface Area}$$
$$= \pi r^2 + \pi r^2 + 2\pi rh$$
$$= 2\pi r^2 + 2\pi rh$$
$$= 2\pi r(r+h)$$

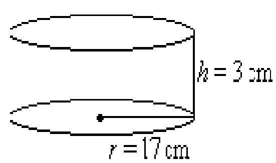
Therefore, the total surface area (*TSA*) of a cylinder with radius *r* and height *h* is given by

$. /0 1 2

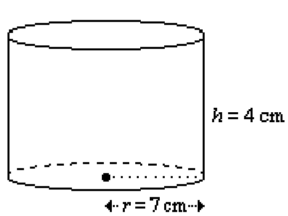
Find the total surface area of a cylindrical tin of radius 17 cm and height 3 cm.

Solution:

$$TSA = 2\pi r(r+h)$$
$$= 2\times 3.142\times 17(17+3)$$
$$= 2\times 3.142\times 17\times 20$$
$$= 2136.56$$



Find the area of the curved surface of a cylindrical tin with radius 7 cm and height 4 cm.



$$CSA = 2\pi rh$$
$$= 2\times\frac{22}{7}\times 7\times 4$$
$$= 176$$

So, the curved surface area is 176 cm^2.

The teacher may start with a question and answer session asking questions of the following type:

- What do you think "volume" mean?
- What is a prism? A cylinder?

Volume is the amount of space occupied by a three-dimensional object; and it is measured in cubic units. The most frequently used units of volume are mm^3, cm^3 and m^3.

A solid with parallel sides is called a **prism**. For example, solids such as a cube or a rectangular box are prisms.

The Volume V, of a solid is given by:

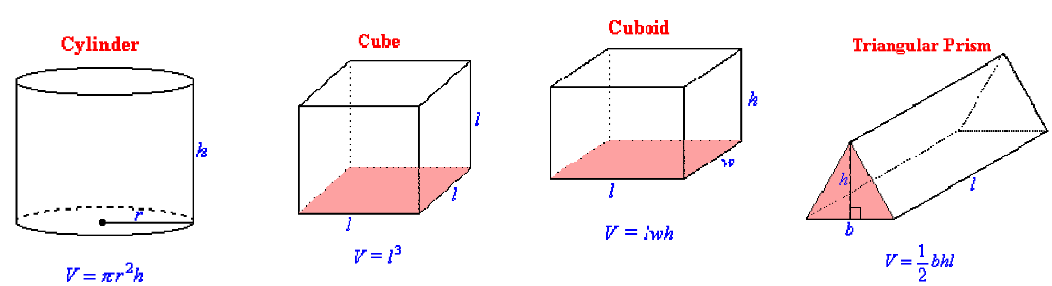
Volume = Area of base x Height

$V = Ah,$

where A is the area of the base (cross-section) and h is the height

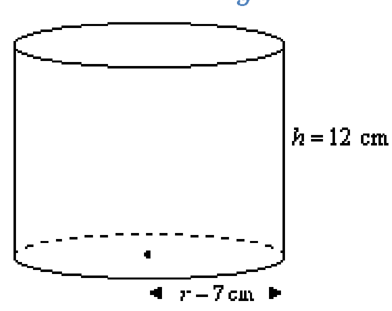
Since Grade 9 students are already familiar with the volumes of some solids in lower grades, the teacher shall use Q & A or independent practice to make students recall some of the formulae given below. The examples also could be given as independent practice recalling questions.

The volumes of the following solids are often required to solve real world problems involving quantity, capacity, mass and strength of materials including liquids.



Find the volume of a cylindrical canister with radius 7 cm and height 12 cm.

Solution:



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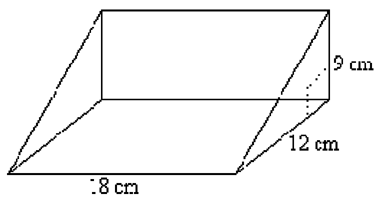
$V = \quad r^2h$

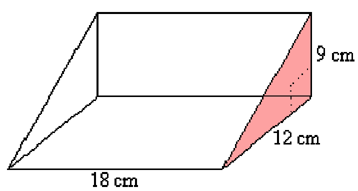
$= \times (7cm)^2 \times 12$ cm

$= 588 \quad cm^3$

Find the volume of the triangular prism shown in the diagram.

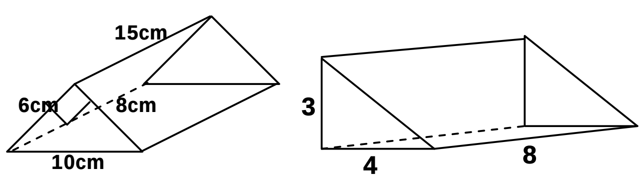


$$V = Al$$
$$= \frac{1}{2}bhl$$
$$= \frac{1}{2} \times 12 \times 9 \times 18$$
$$= 972$$



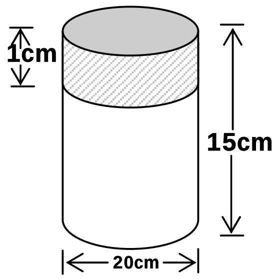
Practice Exercises

1. Calculate the volume and surface area of the solids shown in the figure below.



2. Calculate the total surface area and volume of a cylinder whose

a) Height is 5 cm and base radius 4 cm
b) Height 5 cm and base diameter 6 cm



3. The figure below shows a coffee can with a plastic lid. Calculate the surface area of the plastic lid. What could be the volume of the can below the lid?
4. A cylindrical water tanker has the same length of height as its diameter. What are the volume and total surface area of such tanker?

Practice Exercises

1. Determine the domain and range of the following functions
   A. f(x) = |x|
   B. $f(x) = x^z$
   C. $f(x) = \sqrt{x}$
   D. f(x) = x
2. If f = {(1, 3), (2, 4), (5, 7), (8, 9)} and g = {(1, 4), (2, 0), (3, 5), (8, 1)} then find
   i. The domain of:
      A. f. g
      B. 3f
      C. -2g
      D. $\frac{f}{g}$
   ii. The ordered pairs for:
      A. f. g
      B. 3f
      C. -2g
      D. $\frac{f}{g}$

3. What is the domain of the following functions?
   a) $f(x) = \sqrt{x} + \sqrt{6-x}$.
   b) $f(x) = \begin{cases} 3x-1, & if \quad x > 3 \\ x^2-2, & if \quad -2 \le x \le 3 \\ 2x+3, & if \quad x < -2 \end{cases}$
   c) $f(x) = \frac{1}{x-1} + \frac{1}{x+1}$.

! "

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!

- Determine the variance of a given statistical data
- Find the standard deviation of a given data

Suggested ways of teaching this topic: Q &A and Guided Discovery

Starter Activities

The teacher could start with the revision of previous lessons on the measures of Central Tendency – Mean, Median and Mode using Q & A, asking questions like:

- What is statistics?
- What do we mean by population? Sample?
- What are the measures of Central Tendency?
- How can we find each measure of central Tendency? Show the steps using examples.

Expected Replies

A population is the whole set of items from which a data sample can be drawn; so a sample is only a portion of the population data. Features of a sample are described by statistics.

Statistical methods enable us to arrange, analyze and interpret the sample data obtained from a population. We gather sample data when it is impractical to analyze the population data as a smaller sample often allows us to gain a better understanding of the population without doing too much work or wasting precious time.

Measures of Central Tendency

We make inferences about a population from a sample set of observed values by finding the mean, median and mode. The mean, median and mode are collectively known as **measures of central tendency**

Mean

The **mean** (or average) of a set of values is defined as the sum of all the values divided by the number of values. That is:

$$\text{Mean} = \frac{\text{Sum of all values}}{\text{Number of values}}$$

Symbolically, $\bar{x} = \frac{\sum x}{n}$

where $\bar{x}$ (read as 'x bar') is the mean of the set of x values,
$\sum x$ is the sum of all the x values, and
n is the number of x values.

The marks of five candidates in a mathematics test with a maximum possible mark of 20 are given below:

14; 12; 18; 17; 13.

Find the mean value.

Solution:

$$\bar{x} = \frac{\sum x}{n}$$
$$= \frac{14+12+18+17+13}{5}$$
$$= \frac{74}{5}$$
$$= 14.8$$

So, the mean mark $\bar{x}$ = is 14.8

Median

The **median** is the middle value of the data set arranged in ascending order of magnitude.

Example: The marks of five candidates in a geography test for which the maximum possible mark was 20 are given below:

18;17;15;14;19

Find the median mark.

Solution: Arrange the marks in ascending order of magnitude:

14, 15, 17, 18, 19

The third value, 17, is the middle one in this arrangement. So, median = 17

Note:

$$\text{The number of values, } n \text{, in the data set} = 5$$

$$\text{Median} = \frac{1}{2}(5+1) \text{ th value}$$
$$= \text{3rd value}$$
$$= 17$$

In general:

$$\text{Median} = \frac{1}{2}(n-1) \text{ th value, where } n \text{ is the number of data values in the sample.}$$

If the number of values in the data set is even, then the median is the average of the two middle values.

Find the median of the following scores:

10 16 14 19 8 11

Arrange the values in ascending order of magnitude:

8, 10, 11, 14, 16, 19

There are 6 values in the data set. Therefore n = 6

$$\text{Now, median} = \left(\frac{n+1}{2}\right) \text{th value where } n = 6$$
$$= \left(\frac{6+1}{2}\right)$$
$$= \frac{7}{2}$$
$$= 3.5\text{th value}$$

The third and fourth values, 11 and 14, are in the middle. That is, there is no one middle value.

$$\therefore \text{Median} = \frac{11+14}{2}$$
$$= \frac{25}{2}$$
$$= 12.5$$

***Note**: Half of the values in the data set lie below the median and half lie above the median.*

Mode

The **mode** is the value (or values) that occur most often.

The marks awarded to seven pupils for an assignment were as follows:

18; 14; 18; 15; 12; 19; 18

A. Find the median mark.
B. State the mode.

a) Arrange the marks in ascending order of magnitude:

12, 14, 15, 18 ,18, 18, 19

$$\text{Now, median} = \left(\frac{n+1}{2}\right)\text{th value where } n = 7$$
$$= \left(\frac{7+1}{2}\right)\text{th value}$$
$$= \frac{8}{2}\text{th value}$$
$$= 4\text{th value}$$
$$= 18$$

Note: *The fourth mark, 18, is the middle data value in this arrangement.*

$$\therefore \text{ Median} = 18 \qquad \{\because 18 \text{ is the middle data value}\}$$

b) 18 is the mark that occurs most often (It is repeated 3 times).

After a complete revision of the measures of central tendency, especially, how to determine the mean, the discussions on Variance and Standard Deviation shall continue using guided discovery method, where the teacher leads the students towards the formulae.

3 4

Standard Deviation and Variance

The variance and the closely-related standard deviation are measures of how spread out a distribution is. In other words, they are measures of variability or dispersion. The variance is computed as the average squared deviation of each number from its mean. For example, for the numbers 1, 2, and 3, the mean is 2 and the variance is:

$$\sigma^2 = \frac{(1-2)^2 + (2-2)^2 + (3-2)^2}{3} = 0.667$$

The formula (in summation notation) for the variance in a population is

$$\sigma^2 = \frac{\sum (X-\mu)^2}{N}$$

Where   is the mean and N is the number of scores.

When the variance is computed in a sample, the statistic

$$S^2 = \frac{\sum (X-M)^2}{N}$$

(where M is the mean of the sample) can be used. S^2 is a biased estimate of ^{2}.

However, by far the most common formula for computing variance in a sample is:

$$s^2 = \frac{\sum (X-M)^2}{N-1}$$

which gives an unbiased estimate of ^{2}.

Since samples are usually used to estimate parameters, s^2 is the most commonly used measure of variance. Calculating the variance is an important part of many statistical applications and analyses. It is the first step in calculating the standard deviation.

Standard Deviation

The standard deviation formula is very simple: it is the square root of the variance. It is the most commonly used measure of spread. The standard deviation has proven to be an extremely useful measure of spread in part because it is mathematically controllable. Many formulas in inferential statistics use the standard deviation.

Example: Find the variance and standard deviation, to one decimal place, of population function whose distribution is given below.

Values	1	3	5	7	9
Frequency	2	3	2	3	2

Solution:

Step 1) $\bar{x} = \dfrac{2(1)+3(3)+2(5)+3(7)+2(9)}{2+3+2+3+2} = \dfrac{60}{12} = 5$

	Value	1	3	5	7	9
Step 2)	$x_i - \bar{x}$	1-5	3-5	5-5	7-5	9-5
Step 3)	$(x_i - \bar{x})^2$	$(-4)^2$	$(-2)^2$	$(0)^2$	$(2)^2$	$(4)^2$
	Frequency	2	3	2	3	2
Step 4)	$fi\,(x_i - \bar{x})^2$	$2(-4)^2$	$3(-2)^2$	$2(0)^2$	$3(2)^2$	$2(4)^2$

Step 5) Variance = $\delta^2 = \dfrac{2(16)+3(4)+2(0)+3(4)+2(16)}{12}$

$$= \frac{32+12+0+12+32}{12} = \frac{88}{12} \approx 7.3$$

Step 6) Standard deviation = $\delta = \sqrt{7.3} \approx 2.7$

Example: Find the variance and standard deviation to one decimal place, of the population function whose distribution is given below.

Values	1	2	3	4	5
Frequency	3	3	1	1	4

Solution:

Step 1) $\bar{x} = \frac{3(1)+3(2)+1(3)+1(4)+4(5)}{3+3+1+1+4} = \frac{3+6+3+4+20}{12} = \frac{36}{12} = 3$

	Value	1	2	3	4	5
Step 2)	$x_i - \bar{x}$	1-3	2-3	3 - 3	4 - 3	5-3
Step 3)	$(x_i - \bar{x})^2$	$(-2)^2$	$(-1)^2$	$(0)^2$	$(1)^2$	$(2)^2$
	Frequency	3	3	1	1	4
Step 4)	$f_i (x_i - \bar{x})^2$	$3(-2)^2$	$3(-1)^2$	$1(0)^2$	$1(1)^2$	$4(2)^2$

Step 5) Variance = $\delta^2 = \frac{3(4)+3(1)+1(0)+1(1)+4(4)}{3+3+1+1+4}$

$$= \frac{12+3+0+1+16}{12} \frac{32}{12} \approx 2.7$$

Step 6) The standard deviation

$\delta = \sqrt{2.7} \approx 1.6$ to one decimal place.

Practice Exercises

1. For the numbers 1, 2, 4, 4, 5, 8; find

    i) their mean, median and mode
    ii) the range, variance and standard deviation

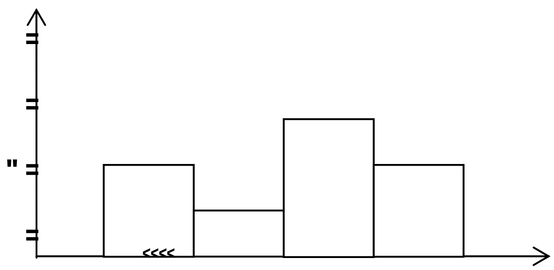
2. Given the following distribution table, find

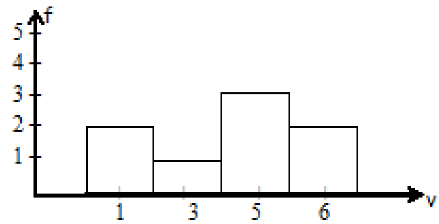
    i) the measures of central tendency
    ii) the measures of dispersion

iii) the histogram that represents the distribution

v	2	4	6	7	10
f	6	2	3	6	1

3. Use the given histogram to find the mean, median, mode, range, mean deviation, variance and standard deviation of the population function.





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- explain the properties of variance and standard deviation
- use the properties of variance and standard deviation to solve related problems

Suggested ways of teaching this topic: Brainstorming and Guided Discovery

Starter Activities

The teacher might start with the following brainstorming questions to revise the previous lesson.

- What is Standard deviation?
- What do we mean by deviation?
- "What is the Variance?"
- How can we find Variance of a given data? The Standard deviation?

Expected Responses

The Standard Deviation is a measure of how spreads out the numbers are.

Its symbol is (the greek letter sigma)

The formula is easy: it is the **square root** of the **Variance.**

Deviation just means how far from the normal

The Variance is defined as: The average of the **squared** differences from the Mean.

To calculate the variance, we follow the following steps:

- Work out the Mean (the simple average of the numbers)
- Then for each number: subtract the Mean and square the result (the *squared difference*).
- Then work out the average of those squared differences. (Why Square?)

Example: Given the population function 2, 4, 1, 5, find the variance and standard deviation. Then add 3 to each data item, and find the variance and standard deviation of the resulting population function compare the variance values and standard deviation values.

Solution: Data items: 2, 4, 1, 5

$$\bar{x} = \frac{2+4+1+5}{4} = \frac{12}{4} = 3$$

$$\delta^2 = \frac{(2-3)^2 + (4-3)^2 + (1-3)^2 + (5-3)^2}{4}$$

$$= \frac{1+1+4+4}{4} = \frac{10}{4} \quad = \quad 2.5$$

$$\delta = \sqrt{2.5}$$

Old data items: 2, 4, 1, 5

New data items: 5, 7, 4, 8

$$\text{New } \bar{x} = \frac{5+7+4+8}{4} = \frac{24}{4} = 6$$

$$\text{New}\delta^2 = \frac{(5-6)^2 + (7-6)^2 + (4-6)^2 + (8-6)^2}{4}$$

$$= \frac{1+1+4+4}{4} = \frac{10}{4} = 2.5$$

$$\delta = \sqrt{2.5}$$

Thus, the new mean is 3 more than the old mean. Both variance and standard deviations are the same.

- At this stage the teacher should ask students: "What can we conclude from this example?" The answer is the following property.

Property 1

If a constant c is added to each value of a population function, then the new variance is the same as that of the old variance. The new standard deviation is also the same as that of the old standard deviation.

- The teacher could ask students to prove this property.

Proof of Property 1

Consider the data items:

$x_1, x_2, x_3, \ldots x_n$ having mean $\bar{x}$.

Add c to each data item: $x_1 + c, x_2 + c, x_3 + c, \ldots x_n + c$ and the new mean $\bar{x}' = \bar{x} + c$

New δ^2

$$= \frac{(x_1 + c - (\bar{x} + c))^2 + (x_2 + c - (\bar{x} + c))^2 + ... + (x_n + c - (\bar{x} + c))^2}{n}$$

$$= \frac{(x_1 + c - \bar{x} + c)^2 + (x_2 + c - \bar{x} - c)^2 + ... + (x_n + c - \bar{x} + c)^2}{n}$$

$= (x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \ldots + (x_n - \bar{x})^2$

$=$ old δ^2

Note: If the values are equal, the square root of the variance will be equal. Therefore, the standard deviations are also equal.

Example: Given the population function, 2, 1, 4, 5, find the mean, variance and standard deviation. Then multiply each data item by 3 and find the new mean, variance and standard deviation. Compare the old and new mean, variance and standard deviation:

Solution: Old data items: 2, 1, 4, 5

$$\bar{x} = \frac{2+1+4+5}{4} = \frac{12}{4} = 3$$

$$\delta^2 = \frac{(2-3)^2 + (1-3)^2 + (4-3)^2 + (5-3)^2}{4}$$

$$= \frac{1+4+1+4}{4}$$

$$= \frac{10}{4} = 2.5$$

$$\delta = \sqrt{2.5}$$

New data items: 6, 3, 12, 15

New mean = $\bar{x} = \frac{6+3+12+15}{4} = \frac{36}{4} = 9$

New variance = $\delta^2 = \frac{(6-9)^2 + (3-9)^2 + (12-9)^2 + (15-19)^2}{4}$

$$= \frac{(-3)^2 + (-6)^2 + (3)^2 + (6)^2}{4}$$

$= \frac{9+36+9+36}{4} = \frac{90}{4} = 22.5$

$\delta = \sqrt{22.5}$

The new mean = $\overline{x}'$ = 9 = 3 $\times$ 3 = $\overline{x}' \times 3$

The new variance = 22.5 = 9 $\times$ 2.5 = $3^2 \times 2.5$

= $3^2 \times$ the old variance.

The new standard deviation = $\sqrt{22.5} = \sqrt{9 \times 2.5} = \sqrt{3^2 \times 2.5}$

$= |3|\sqrt{2.5}$

$= |3| \times$ the old st. deviation

- At this stage the teacher should ask students: "What can we conclude from this example?" The answer is the following property.

Property 2

If each data item of a population function is multiplied by a constant c, the new variance is c^2 times the old variance. The new standard deviation is |c| times the old standard deviation.

- The teacher could ask students to proof this property for the general case

Proof of Property 2

Consider the population function x_1, x_2, x_3, . . . x_n whose mean is $\overline{x}$ and variance δ^2.

New data items: cx_1, cx_2, cx_3, . . . cx_n having new mean = $c\overline{x}$.

New variance = $\frac{(cx_1 - c\overline{x})^2 \ (cx_2 - c\overline{x})^2 + ... + (cx_n - c\overline{x})^2}{n}$

$= \frac{c^2 (x_1 - \overline{x})^2 + (x_2 - \overline{x})^2 + ... + (x_n - \overline{x})^2}{n} = = c^2 \delta^2$

= $c^2 \times$ the old variance

New standard deviation = $\sqrt{c^2 \delta^2} = |c|\delta$

Concluding Activities

The teacher shall make sure that the following points are well taken and summarized by the students.

If all the values of a population are increased by a constant c then, the mean is also increased by c while the standard deviation remains unchanged.
If all the values of a population are multiplied by a constant c then,

i) The new mean is c $\bar{x}$ the old mean

ii) The new standard deviation is |c| $\bar{x}$ the old standard deviation

Practice Exercises

1. What is the standard deviation for the numbers: 75, 83, 96, 100, 121 and 125?
2. Find the variance and standard deviation of the given population function:- 3, 5, -1, 3. Then subtract 2 from each data item, and find the variance and standard deviation of the new data items. Compare the old and new variance values and standard deviation. What can you conclude?
3. Given the population function, 2,1,3,2, find the mean, variance and standard deviation. Then multiply each data item by – 2 and find the new mean, variance and standard deviation. Compare the old and new mean, variance and standard deviation.
4. Use the given frequency distribution table to find the mean, variance and standard deviation of the population function.

Value	-2	0	2	3	4
Frequency	1	2	2	2	3

5. What is the standard deviation of the first 10 natural numbers (1 to 10)?
6. The standard deviation of the numbers 3, 8, 12, 17 and 25 is 7.56 correct to 2 decimal places. What will be the new standard deviation if every value is:

   a) Increased by 4?
   b) Decreased by 2?
   c) Multiplied by 4?
   d) Halved?

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- Determine the set of all possible outcomes
- Find the probability of different events
- Apply tree diagrams to determine outcomes of experiments

Suggested Methods of Teaching The Topic: Teacher explanation, small group activities

Starter Activities

The teacher could start with recalling Q & A session, asking the following:

- What do we mean by sample space or possibility set?
- What is an event? Give examples for each

Expected Answers

The sample space (*S*) of an experiment is the set of all possible outcomes of any trial of the experiment to be conducted.

An event (*E*) is a subset of the sample space. That is, an event is a subset of all possible outcomes. We refer to this subset of outcomes as **favorable outcomes**.

For example, the sample space for an experiment of tossing a fair coin is $S = \{H, T\}$, and the the two possible outcomes are the events $E_1 = \{H\}$ and $E_2 = \{T\}$.

Note that E_1 is the event that 'a head falls' and E_2 is the event that a tail falls'.

Lesson Notes

Probability of an Event

The probability of event *E* occurring is given by

$$\Pr(E) = \frac{\text{Number of outcomes in event } E}{\text{Number of outcomes in sample space } S}$$

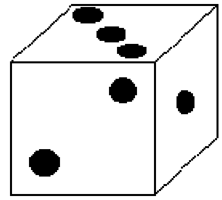
This is often written as:

$$\Pr(E) = \frac{n(E)}{n(S)}$$

This result holds only if the outcomes of an experiment are equally likely.

Note: *The events are denoted by capital letters A, B, C, D, E, ...*

A die is rolled. Find:



a) the sample space for this experiment
b) the probability of obtaining an even number
c) the probability of obtaining a prime number

Solution:

a. $S = \{1, 2, 3, 4, 5, 6\}$

b. Let A be the event that an even number is obtained.

$\therefore A = \{2, 4, 5\}$

$$\Pr(A) = \frac{n(A)}{n(S)} = \frac{3}{6} = \frac{1}{2}$$

c. Let B be the event that a prime number is obtained.

$\therefore B = \{2, 3, 5\}$

$$\Pr(B) = \frac{n(B)}{n(S)}$$
$$= \frac{3}{6}$$
$$= \frac{1}{2}$$

Range of Probability

If an event is **impossible**, its probability is 0. If an event is **certain** to occur, its probability is 1. The probability of any other event is between these two values. That is:

- $\Pr(\text{impossible event}) = 0$
- $\Pr(\text{certain event}) = 1$
- If A is any event, then $0 \le \Pr(A) \le 1$.

Event:	Impossible Event	Even Chance	Certain
Probability:	0	$\frac{1}{2}$	1

A die is rolled. Find the probability of obtaining:

a) A 7

b) A number less than or equal to 6

Solution:

$S = \{1, 2, 3, 4, 5, 6\}$

a. It is impossible to obtain a 7

$\therefore \quad \Pr(a\ 7) = 0$

b. Let A be the event that a number less than or equal to 6 is obtained. Then:

$A = \{1, 2, 3, 4, 5, 6\}$

$$\text{Now, } \Pr(A) = \frac{\text{Number of outcomes in } A}{\text{Number of outcomes in } S}$$
$$= \frac{6}{6}$$
$$= 1$$

Note:

- *It is certain that event A will occur as it contains all 6 possible outcomes.*
- *7 is not an outcome of rolling a die as it is not possible.*

A pack of 52 playing cards consists of four suits, i.e. clubs, spades, diamonds and hearts. Each suit has 13 cards which are the 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, king and the ace card. Clubs and spades are of black color whereas diamonds and hearts are of red color. So, there are 26 red cards and 26 black cards. Find the probability of drawing from a well-shuffled pack of cards:

A) A black card
B) The king of diamonds
C) A jack

a. A pack of 52 cards has 26 black cards.

$$\therefore \Pr(\text{a black card}) = \frac{26}{52}$$
$$= \frac{1}{2}$$

b. A pack of 52 cards has 1 king of diamonds.

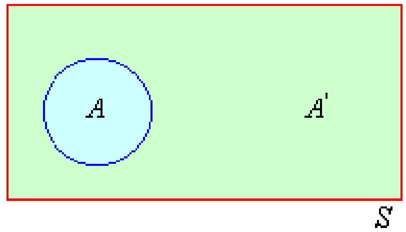
$$\therefore \Pr(\text{the king of diamonds}) = \frac{1}{52}$$

c. A pack of 52 cards has 4 jacks.

$$\therefore \Pr(\text{a jack}) = \frac{4}{52}$$
$$= \frac{1}{13}$$

Complement of Event *A*

A' is the **complement** of event *A*. It contains all of the elements in the sample space*S* that are not included in *A*.



It is certain that either *A* or A' must occur. So, it follows that: For any event *A* and its complement A':

$$\Pr(A) + \Pr(A') = 1$$

The probability that a train will be late is $\frac{1}{100}$. Find the probability that it will be on time.

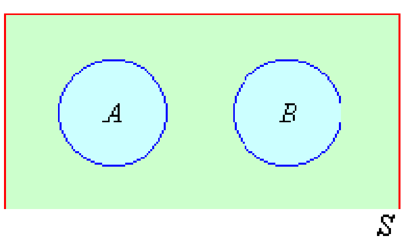
Let # be the event that a train will be late. Then #' is the event that it will be on time.

$$\Pr(A) + \Pr(A') = 1$$

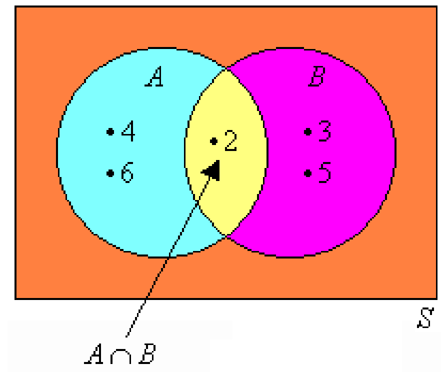
$$\therefore \frac{1}{100} + \Pr(A') = 1$$

$$\therefore \Pr(A') = 1 - \frac{1}{100} = \frac{100-1}{100} = \frac{99}{100}$$

A and *B* are said to be **mutually exclusive events** if they do not overlap. This means that *A* and *B* are mutually exclusive events such that if *A* occurs then *B* is excluded or if *B* occurs then *A* is excluded. That is, *A* and *B* cannot occur together.



$S = \{1, 2, 3, 4, 5, 6\}$



Let the events be defined as follows:

A = the event that an even number is obtained; and
B = the event that a prime number is obtained.

$\therefore\ A = \{2, 4, 6\}$

$\quad B = \{2, 3, 5\}$

$A \cap B = \{2\}$

$A \cup B = \{2, 3, 4, 5, 6\}$

Note: *Mutually exclusive events have no sample points in common.*

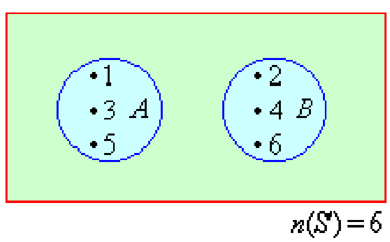
Consider the experiment of throwing a die. Let A be the event that an odd number is obtained and B be the event that an even number is obtained. Then:

$A = \{1, 3, 5\}$

$B = \{2, 4, 6\}$

$\therefore\ A \cap B = \phi$ $\qquad$ $\{\cap$ stands for 'intersection' or 'and'$\}$

That is, A and B have no elements (sample points) in common. Hence A and B are mutually exclusive events, as shown in the following Venn diagram.



$$\text{Now, } \Pr(A \cap B) = \Pr(A \text{ and } B) = \frac{\text{Number of elements in } A \text{ and } B}{\text{Number of elements in } S}$$

$$= \frac{0}{6}$$

$$= 0$$

Addition Law of Probabilities: For the example under consideration:

$$\Pr(A) = \frac{\text{Number of elements in } A}{\text{Number of elements in } S} = \frac{3}{6} = \frac{1}{2}$$

$$\Pr(B) = \frac{\text{Number of elements in } B}{\text{Number of elements in } S} = \frac{3}{6} = \frac{1}{2}$$

$$\Pr(A \text{ or } B) = \frac{\text{Number of elements in } A \text{ or } B}{\text{Number of elements in } S} = \frac{6}{6} = 1 \qquad \ldots(1)$$

$$\text{Also, } \Pr(A) + \Pr(B) = \frac{1}{2} + \frac{1}{2} = 1 \qquad \ldots(2)$$

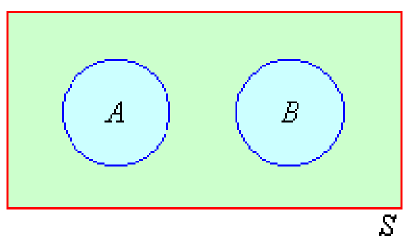
From (1) and (2), we obtain:

$\Pr(A \text{ or } B) = \Pr(A) + \Pr(B)$

Note: $\Pr(A \text{ or } B)$ is also denoted by $\Pr(A \cup B)$.

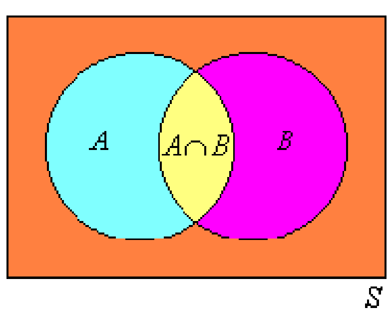
In general:

1. If A and B are mutually exclusive events, then:
   $\Pr(A \cup B) = \Pr(A) + \Pr(B)$



If A and B overlap then,

2. If A and B overlap, then:
   $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$



Consider the experiment of throwing a die. As usual:

$$\begin{aligned}\Pr(A \cup B) &= \frac{\text{Number of elements in } A \cup B}{\text{Number of elements in } S} \\ &= \frac{5}{6} \qquad \ldots(1)\end{aligned}$$

$$\Pr(A) = \frac{3}{6}$$

$$\Pr(B) = \frac{3}{6}$$

$$\Pr(A \cap B) = \frac{1}{6}$$

We notice that:

$$\begin{aligned}\Pr(A) + \Pr(B) - \Pr(A \cap B) &= \frac{3}{6} + \frac{3}{6} - \frac{1}{6} \\ &= \frac{5}{6} \qquad \ldots(2)\end{aligned}$$

From (1) and (2), we obtain:

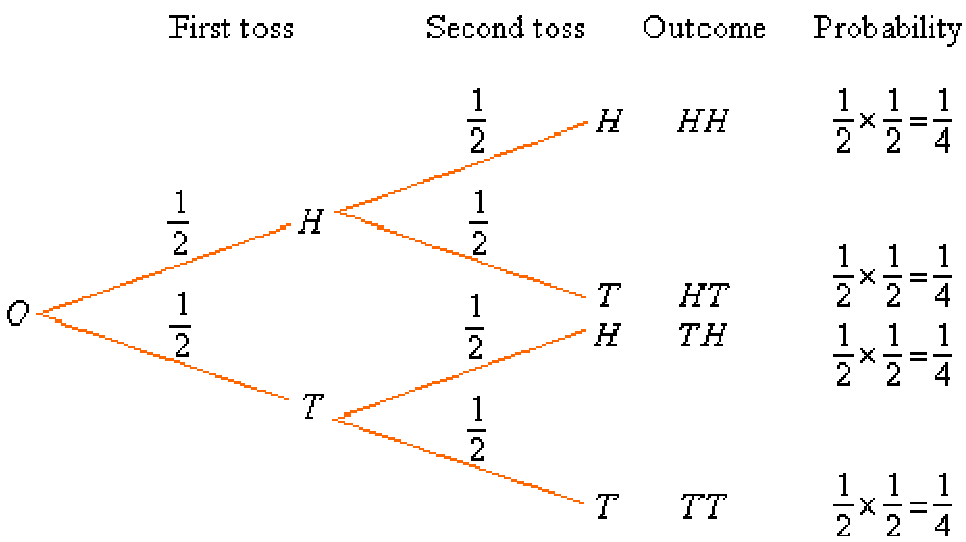
$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$$

This is called the **addition law of probabilities**.

A die is rolled. If $A = \{\text{greater than } 3\}$ and $B = \{\text{prime}\}$, find $\Pr(A \text{ or } B)$.

$S = \{1, 2, 3, 4, 5, 6\}$
$A = \{4, 5, 6\}$
$B = \{2, 3, 5\}$
$\therefore\ A \cap B = \{5\}$

$$\begin{aligned}\text{Now, } \Pr(A \cup B) &= \Pr(A) + \Pr(B) - \Pr(A \cap B) \\ &= \frac{3}{6} + \frac{3}{6} - \frac{1}{6} \\ &= \frac{5}{6}\end{aligned}$$



So, far, we have mainly considered simple probability experiments such as tossing a coin or throwing a die. Usually, probability experiments are more complicated. For example, tossing two different coins, tossing a coin and throwing a die, or tossing the same coin three times etc.

Recall the experiment of tossing a coin twice where we are interested in the number of heads. The tree diagram is shown below.

Note that the tree diagram representation of this experiment involves two parts, 'the first toss of the coin' and 'the second toss of the coin'. Experiments that have two parts can be represented in tabular form.

For example, the following table uses rows to represent 'the first toss of the coin' and columns to represent 'the second toss of the coin'. The experiment's outcomes are shown in the bottom right-hand corner of the table where the rows and columns intersect.

		Second toss	
		H	T
First toss	H	HH	HT
	T	TH	TT

That is, $S = \{HH, HT, TH, TT\}$

Clearly, $\Pr(\text{two heads}) = \frac{1}{4}$

$\Pr(\text{one head}) = \frac{2}{4} = \frac{1}{2}$

$\Pr(\text{no heads}) = \frac{1}{4}$

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A = the numbers facing upwards on the two dice are the same
B = the sum of the numbers facing upwards on the two dice is 10
C = the sum of the numbers facing upwards on the two dice is 13

Find:

a. $\Pr(A)$ b. $\Pr(B)$ c. $\Pr(A \cup B)$ d. $\Pr(A \cap B)$ e. $\Pr(C)$

Solution:

Two dice are thrown. So, there are 36 elements in the sample space as shown in the table below.

		Second throw					
		1	2	3	4	5	6
First throw	1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
	2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
	3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
	4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
	5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
	6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

$$\therefore S = \left\{ \begin{array}{l} (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), \\ (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), \\ (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), \\ (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), \\ (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), \\ (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \end{array} \right\}$$

$$n(S) = 36$$

a. $A = \{(1,\ 1),\ (2,\ 2),\ (3,\ 3),\ (4,\ 4),\ (5,\ 5),\ (6,\ 6)\}$

$$\Pr(A) = \frac{n(A)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

b. $B = \{(6,4),\ (5,5),\ (4,6)\}$

$$\Pr(B) = \frac{n(B)}{n(S)} = \frac{3}{36} = \frac{1}{12}$$

c. $A \cup B = \{(1,\ 1),\ (2,\ 2),\ (3,\ 3),\ (4,\ 4),\ (5,\ 5),\ (6,\ 6),\ (6,4),\ (4,6)\}$

$$\Pr(A \cup B) = \frac{n(A \cup B)}{n(S)} = \frac{8}{36} = \frac{2}{9}$$

d. $A \cap B = \{(5,5)\}$

$$\Pr(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{1}{36}$$

e. $C = \phi$

$$\Pr(C) = \frac{n(C)}{n(S)} = \frac{0}{36} = 0$$

Practice Exercises

1. A lady has 3 dresses, 4 skirts and 5 blouses in her wardrobe. She decides to pack 2 dresses, 3 skirts and 3 blouses in her suitcase for her holiday. How many different choices can she make?
2. A group of 12 people are waiting to board a taxi. The first taxi to arrive can only transport 4 of them. The following taxi can transport 5 of them and the third taxi can seat the remainder. In how many ways can the group be transported?
3. A brother and sister and 5 other people are seated at random in a row at a concert. What is the probability that the brother and sister sit next to each other?
4. A student buys sweets at the shop. On display are 3 chocolates, 4 toffees, 3 lollipops and 2 jellybeans, all of them different. In how many ways can the student choose 6 sweets of which 2 are chocolates and 1 is a toffee?
5. How many four-digit numbers can be formed from the digits 2, 3, 4, 5, 6, and 7

   a) Without repetition?
   b) With repetition?

6. How many ways are there to deal a five -card hand consisting of three eight's and two sevens from a full deck of 52cards?